AP Calculus AB · Practice Worksheet · tutorinmontreal.ca

Limits and Continuity

Name: ______________________________ Date: ______________

Concepts

- Can Change Occur at an Instant? (theory)
- Defining Limits and Using Limit Notation
- Estimating Limit Values from Graphs
- Estimating Limit Values from Tables (theory)
- Determining Limits Using Algebraic Properties
- Determining Limits Using Algebraic Manipulation
- Selecting Procedures for Determining Limits (theory)
- Determining Limits Using the Squeeze Theorem (theory)
- Connecting Multiple Representations of Limits (theory)
- Exploring Types of Discontinuities
- Defining Continuity at a Point
- Confirming Continuity over an Interval
- Removing Discontinuities (theory)
- Connecting Infinite Limits and Vertical Asymptotes
- Connecting Limits at Infinity and Horizontal Asymptotes
- Working with the Intermediate Value Theorem

Can Change Occur at an Instant?

1. A drone rises from a launch pad. Its height above the pad, in metres, is $h(t) = 40t - 5t^2$, where t is in seconds.

a) Find the average rate of change of h over the interval $[1, 3]$, with units.

b) Compute the average rate of change over $[1, 1.1]$, over $[1, 1.01]$ and over $[1, 1.001]$.

c) Based on part (b), what single number does the drone's velocity at the *instant* $t = 1$ appear to be? Explain in one sentence why no single one of the intervals in part (b) is enough to answer this.

Defining Limits and Using Limit Notation

2. A function g is defined by $g(x) = 2x + 5$ for every $x \neq 4$, and $g(4) = 1$.

a) Write, in correct limit notation, the statement "the values of $g(x)$ get arbitrarily close to 13 as x gets sufficiently close to 4."

b) Evaluate that limit.

c) Explain in one sentence why the fact that $g(4) = 1$ does not change your answer to part (b).

Estimating Limit Values from Graphs

3. The graph of f consists of three pieces: a line segment rising from $(-2, 1)$ to an open circle at $(2, 5)$; a solid dot at $(2, 2)$; and a horizontal segment from an open circle at $(2, 3)$ across to $(6, 3)$.

a) Find $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$.

b) Does $\lim_{x\to 2} f(x)$ exist? Justify your answer.

c) State $f(2)$.

Estimating Limit Values from Tables

4. A student tabulates a function f near $x = 2$ and records $f(1.9) = 0.2516$, $f(1.99) = 0.2502$, $f(1.999) = 0.2500$, $f(2.001) = 0.2500$, $f(2.01) = 0.2498$, $f(2.1) = 0.2485$. The value $f(2)$ is undefined.

a) Estimate $\lim_{x \to 2} f(x)$, and say which entries in the list justify treating the limit as two-sided.

b) Explain in one or two sentences why a table like this can only *estimate* the limit and can never prove its value.

Determining Limits Using Algebraic Properties

5. Suppose $\lim_{x \to 3} f(x) = -2$ and $\lim_{x \to 3} g(x) = 5$. Find each limit, naming the limit property you use at the decisive step.

a) $\lim_{x \to 3} \big(3f(x) - 2g(x)\big)$

b) $\lim_{x \to 3} \big(f(x)\, g(x) + 4\big)$

c) $\lim_{x \to 3} \dfrac{g(x) + 3}{f(x)}$

d) $\lim_{x \to 3} \sqrt{g(x) + 11}$

Determining Limits Using Algebraic Manipulation

6. Evaluate $\lim_{x \to -4} \dfrac{x^2 + x - 12}{x + 4}$. Begin by explaining, in one sentence, why substituting $x = -4$ does not settle this limit.

Selecting Procedures for Determining Limits

7. For each limit, first state which procedure is appropriate — substitution, factoring, or multiplying by a conjugate — and then evaluate it.

a) $\lim_{x \to 2} \dfrac{x^3 - 5x}{x + 1}$

b) $\lim_{x \to 5} \dfrac{x^2 - 25}{x^2 - 3x - 10}$

c) $\lim_{x \to 0} \dfrac{\sqrt{x + 16} - 4}{x}$

Determining Limits Using the Squeeze Theorem

8. Let $f(x) = x^2 \cos\left(\frac{1}{x}\right)$ for $x \neq 0$.

a) Write an inequality of the form $a(x) \leq f(x) \leq b(x)$ that holds for every $x \neq 0$, and say what fact about the cosine function justifies it.

b) Use the squeeze theorem to find $\lim_{x \to 0} f(x)$, stating why the theorem's hypotheses are satisfied.

Connecting Multiple Representations of Limits

9. Let $f(x) = \dfrac{x^2 - 2x - 8}{x - 4}$.

a) Evaluate $\lim_{x \to 4} f(x)$ analytically.

b) The graph of f is a straight line with a single point missing. Give the equation of that line and the coordinates of the missing point.

c) Predict the value of $f(3.999)$, to three decimal places, and say how it supports your answer to part (a).

Exploring Types of Discontinuities

10. Let $f(x) = \dfrac{(x-1)(x+3)}{(x-1)(x-5)}$. Identify every value of x at which f is discontinuous, and classify each discontinuity as removable, jump, or infinite. Justify each classification with a limit statement.

Defining Continuity at a Point

11. Let $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ for $x \neq 2$, and $f(2) = 3$. Verify, one condition at a time, that f is continuous at $x = 2$. Name each of the three conditions as you check it.

Confirming Continuity over an Interval

12. Find every interval on which $f(x) = \dfrac{x+1}{x^2 - x - 6}$ is continuous. Justify your answer by naming the theorem about where rational functions are continuous.

Removing Discontinuities

13. Let

$$f(x) = \begin{cases} \dfrac{2x^2 + 5x + 3}{x+1}, & x \neq -1 \\ k, & x = -1. \end{cases}$$

Find the value of k that makes f continuous at $x = -1$, and explain why exactly one value works.

Connecting Infinite Limits and Vertical Asymptotes

14. Let $f(x) = \dfrac{2x}{x-5}$.

a) Find $\lim\limits_{x \to 5^-} f(x)$ and $\lim\limits_{x \to 5^+} f(x)$, showing the sign reasoning for each.

b) State the equation of the vertical asymptote, and explain how part (a) establishes it.

c) Does $\lim\limits_{x \to 5} f(x)$ exist? Justify your answer.

Connecting Limits at Infinity and Horizontal Asymptotes

15.

a) Evaluate $\lim_{x\to\infty} \dfrac{3x^2-5x+1}{6x^2+4}$ and $\lim_{x\to-\infty} \dfrac{3x^2-5x+1}{6x^2+4}$.

b) Evaluate $\lim_{x\to\infty} \dfrac{2x+7}{x^2-1}$ and $\lim_{x\to-\infty} \dfrac{2x+7}{x^2-1}$.

c) State every horizontal asymptote of each of the two functions above.

Working with the Intermediate Value Theorem

16. Let $f(x) = x^3 - 4x + 1$.

a) Compute $f(0)$ and $f(1)$.

b) Show that there is a number c in the open interval $(0, 1)$ with $f(c) = 0$. Name the theorem you use and verify that each of its hypotheses is satisfied.

Synthesis — drawing on several topics in this unit

17. Let $f(x) = \dfrac{x^2 - 7x + 10}{x^2 - 4x + 4}$.

a) Write f in lowest terms, stating the values of x excluded from its domain.

b) Find $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$, and give the equation of the vertical asymptote.

c) Find $\lim_{x \to \infty} f(x)$ and give the equation of the horizontal asymptote.

d) Classify the discontinuity at $x = 2$, and state every interval on which f is continuous.