CEGEP Linear Algebra · Practice Worksheet · tutorinmontreal.ca

Matrices and Matrix Algebra

Name: ____ Date: __________________

Concepts

- Matrix Notation and Special Matrices
- Addition and Scalar Multiplication
- Matrix Multiplication
- Properties and Failures of Matrix Algebra (theory)
- The Transpose and Symmetric Matrices (theory)
- Diagonal and Triangular Matrices (theory)

Matrix Notation and Special Matrices

1. Let

$$M = \begin{bmatrix} 3 & -1 & 0 & 5 \\ 2 & 4 & -6 & 1 \\ 0 & 7 & 2 & -3 \end{bmatrix}.$$

a) State the size of M, and give the entries m_{23} and m_{32}.

b) Write out in full the 3×3 matrix B whose entries are given by $b_{ij} = 2i - j$.

c) Write down the 2×4 zero matrix and the identity matrix I_4, and give the entries $(I_4)_{33}$ and $(I_4)_{23}$.

Addition and Scalar Multiplication

2. Let

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} -3 & 5 & 1 \\ 6 & 0 & -4 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}.$$

a) Compute $3A - 2B$.

b) Find the matrix X satisfying $A + 2X = B$.

c) Explain in one sentence why $A + C$ is not defined.

Matrix Multiplication

3. Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 5 \\ -2 & 0 \\ 4 & 3 \end{bmatrix}.$$

a) State the sizes of AB and of BA.

b) Compute AB.

c) Compute the single entry $(BA)_{31}$ without computing the rest of BA.

Properties and Failures of Matrix Algebra

4. Matrix algebra keeps most of the rules of ordinary algebra and loses a few. This question is about the ones it loses. Throughout, 0 denotes the 2×2 zero matrix.

a) With $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, compute AB and BA, and say which rule of ordinary algebra this refutes.

b) With $C = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$, compute CD. Which property of the real numbers has just failed?

c) Let $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} 4 & 7 \\ 2 & 3 \end{bmatrix}$. Show that $CE = CF$ even though $E \neq F$ and $C \neq 0$, and name the step of ordinary algebra that is therefore invalid for matrices.

d) Prove or disprove: for all 2×2 matrices A and B, $(A + B)^2 = A^2 + 2AB + B^2$. If it is false, give the correct expansion and say exactly which step of the usual argument breaks.

The Transpose and Symmetric Matrices

5. Let

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 0 & 3 & 5 \end{bmatrix}, \qquad K = \begin{bmatrix} 0 & 5 & -3 \\ -5 & 0 & 2 \\ 3 & -2 & 0 \end{bmatrix}.$$

a) Write A^T and state its size.

b) Compute AA^T and confirm that it is symmetric.

c) Show that K is antisymmetric, and explain why *every* antisymmetric matrix must carry zeros all along its main diagonal.

d) Show that the zero matrix is the only square matrix that is both symmetric and antisymmetric.

Diagonal and Triangular Matrices

6. Let

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 5 & 3 \\ 0 & 0 & -1 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ -3 & 4 & 5 \end{bmatrix}.$$

a) Classify each of D, U, L as diagonal, upper triangular or lower triangular. One of them deserves more than one label; say which, and why.

b) Compute DU and UD. Describe in words what multiplying by a diagonal matrix on the left does, and what it does on the right.

c) Compute U^2, then explain why the product of any two upper triangular 3×3 matrices must again be upper triangular.

Synthesis — drawing on several topics in this set

7. Every square matrix splits into a symmetric part and an antisymmetric part. For a square matrix A, define

$$S = \tfrac{1}{2}\left(A + A^T\right), \qquad K = \tfrac{1}{2}\left(A - A^T\right).$$

a) For $A = \begin{bmatrix} 4 & 2 \\ 6 & -2 \end{bmatrix}$, compute S and K.

b) Verify for this A that S is symmetric, that K is antisymmetric, and that $S + K = A$.

c) Prove that S is symmetric and K is antisymmetric for *every* square matrix A. You may use $(X+Y)^T = X^T + Y^T$, $(cX)^T = cX^T$ and $\left(X^T\right)^T = X$.