CEGEP Linear Algebra · Practice Worksheet · tutorinmontreal.ca

Systems of Linear Equations

Name: ________________________ Date: ______________

Concepts

- Linear Systems and the Augmented Matrix
- Row Reduction to Echelon Form
- Gaussian Elimination
- Gauss-Jordan Elimination
- Interpreting the Type of Solution Set
- Homogeneous Systems (theory)

Linear Systems and the Augmented Matrix

1. Consider the linear system

$$\begin{array}{rcrcrcr} 2x & - & y & + & 3z & = & 1 \\ x & & & + & 4z & = & -1 \\ -3x & + & 5y & & & = & 1 \end{array}$$

a) Write its augmented matrix, using the variable order x, y, z.

b) Write out the system whose augmented matrix is $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 5 \\ 0 & 3 & -1 & -4 \end{array}\right]$, using the variables x_1, x_2, x_3.

c) Verify that $(x, y, z) = (3, 2, -1)$ is a solution of the system above, and say in one sentence what that verification amounts to, row by row, in the augmented matrix.

Row Reduction to Echelon Form

2. Reduce $\begin{bmatrix} 0 & 2 & -4 & 6 \\ 1 & -1 & 3 & 2 \\ 2 & 1 & -1 & 9 \end{bmatrix}$ to a row echelon form, writing each elementary row operation in the form $R_2 \to R_2 - 3R_1$ as you use it. Then state which columns contain the leading entries, and explain why an interchange was needed before any elimination could start.

Gaussian Elimination

3. Solve

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 6 \\ 2x & + & 5y & + & z & = & 11 \\ -x & & & + & 3z & = & -4 \end{array}$$

by *Gaussian elimination*: reduce the augmented matrix to row echelon form only, then finish by back-substitution. Do not carry the reduction as far as reduced row echelon form.

Gauss-Jordan Elimination

4. Solve

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 6 \\ 2x & - & y & + & z & = & 9 \\ x & + & 2y & - & z & = & -1 \end{array}$$

by *Gauss-Jordan elimination*: carry the augmented matrix all the way to reduced row echelon form and read the solution off. Name each row operation.

Interpreting the Type of Solution Set

5. Each matrix below is the reduced row echelon form of the augmented matrix of a system in the unknowns x, y, z. For each one: state how many free variables the system has, classify it as having a unique solution, infinitely many solutions or no solution, and write the solution set explicitly — with a parameter where one is needed.

$$\text{(i)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right] \qquad \text{(ii)} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{(iii)} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Homogeneous Systems

6. Consider the homogeneous system

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 0 \\ 2x & + & 5y & + & z & = & 0 \\ x & + & y & - & 4z & = & 0 \end{array}$$

a) Solve it by Gauss-Jordan elimination and write the solution set with a parameter.

b) State the trivial solution and give one non-trivial solution with integer entries.

c) Explain why *no* homogeneous system can be inconsistent, arguing from what row operations do to the last column of the augmented matrix.

Synthesis — drawing on several topics in this set

7. Over one morning a café serves 100 cups of coffee in three sizes: small (200 mL each), medium (300 mL each) and large (400 mL each). The urn readings show that 28 litres of coffee were poured, and the sachet count shows 180 sugar sachets used, at 1 sachet per small cup, 2 per medium and 3 per large.

- **a)** Let x, y, z be the numbers of small, medium and large cups. Write the system and its augmented matrix, measuring coffee in units of 100 mL.
- **b)** Reduce the matrix to reduced row echelon form and account for what you find in the third row.
- **c)** Classify the solution set and write it with a parameter.
- **d)** The three counts must be non-negative whole numbers. Determine which values of the parameter are admissible, how many sales patterns are possible, and give the two extreme ones.

Want the harder problems and the worked solutions?

The *Challenge Problems* set and the *Answer Keys* for Systems of Linear Equations — with the reasoning written out in full — are available at **tutorinmontreal.ca**.