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The Inverse of a Matrix

Name: ______________________________ Date: ______________

Concepts

- The Inverse of a Square Matrix
- Finding an Inverse by Row Reduction
- Elementary Matrices (theory)
- Solving a System with the Inverse Matrix
- Conditions Equivalent to Invertibility (theory)

The Inverse of a Square Matrix

1. An $n \times n$ matrix A is *invertible* when there is an $n \times n$ matrix A^{-1} with $AA^{-1} = A^{-1}A = I_n$. For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ there is a rule: if the number $ad - bc$ is not zero, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

and if $ad - bc = 0$ the rule produces nothing — part (b) asks you to settle that case yourself.

a) Find A^{-1} for $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, and confirm your answer by computing AA^{-1}.

b) Apply the rule to $B = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$. Then, without using the rule at all, give a non-zero column $\vec{x}$ with $B\vec{x} = \vec{0}$ and say in one sentence why the existence of such an $\vec{x}$ already rules out an inverse for B.

c) A matrix M satisfies $M^{-1} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$. Recover M.

Finding an Inverse by Row Reduction

2. Find A^{-1} for $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix}$ by reducing the augmented array $[\,A \mid I_3\,]$ to $[\,I_3 \mid A^{-1}\,]$. Name each row operation you use, and check one column of your answer by multiplying it back into A.

Elementary Matrices

3. An *elementary matrix* is what you get by performing a single row operation on the identity matrix. Work throughout with 3×3 matrices, and take $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 5 \\ 4 & 1 & 0 \end{bmatrix}$.

a) Write down the elementary matrices E_1, E_2 and E_3 corresponding to $R_3 \to R_3 - 4R_1$, to $R_2 \to \frac{1}{2}R_2$, and to $R_1 \leftrightarrow R_2$.

b) Compute $E_1 A$, and confirm that it is A with that row operation carried out.

c) Write down E_1^{-1}, E_2^{-1} and E_3^{-1}, naming the row operation each one performs, then explain in two sentences why every elementary matrix is invertible and why its inverse is again elementary.

Solving a System with the Inverse Matrix

4. A soap workshop makes three blends, X, Y and Z. One batch of each blend uses the kilograms of olive oil, coconut oil and shea butter recorded in the corresponding column of

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}, \qquad \text{rows: olive oil, coconut oil, shea butter.}$$

If $\vec{x}$ lists the numbers of batches made, then $A\vec{x}$ lists the kilograms consumed, so "a delivery $\vec{b}$ is used up exactly" means $A\vec{x} = \vec{b}$.

a) Verify that $A^{-1} = \begin{bmatrix} 6 & -2 & -1 \\ 1 & 0 & -1 \\ -3 & 1 & 1 \end{bmatrix}$ by computing AA^{-1}.

b) This week's delivery is 9 kg of olive oil, 22 kg of coconut oil and 8 kg of shea butter. Use $\vec{x} = A^{-1}\vec{b}$ to find the batch numbers.

c) Next week's delivery is 7, 18 and 5 kg. Find those batch numbers too.

d) In one sentence, say why computing A^{-1} once is the efficient method here, whereas it would not be for a single delivery.

Conditions Equivalent to Invertibility

5. For an $n \times n$ matrix A the following statements are either all true or all false: A is invertible; the reduced row echelon form of A is I_n; A has n pivots; $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$; $A\vec{x} = \vec{b}$ has exactly one solution for every $\vec{b}$; the columns of A are linearly independent. Apply that list to

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 1 & 3 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

For each matrix decide whether it is invertible, justifying the decision by counting pivots. For each one that is *not* invertible, also give a non-zero $\vec{x}$ sent to $\vec{0}$ by that matrix, and write one of its columns as a combination of the other two, so that two further statements on the list are seen to fail as well. Compute no inverses.

Synthesis — drawing on several topics in this set

6. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

a) Reduce A to I_2 using exactly three row operations, and write down the elementary matrices E_1, E_2, E_3 of those operations, in the order you used them.

b) From $E_3E_2E_1A = I_2$, read off A^{-1} as a product of elementary matrices and multiply it out. Confirm the result against the 2×2 rule.

c) Write A itself as a product of elementary matrices, and explain what this computation illustrates about the statement "A is a product of elementary matrices" as an addition to the invertibility list.