CEGEP Linear Algebra · Practice Worksheet · tutorinmontreal.ca

Determinants

Name: ____ Date: __________________

Concepts

- Determinants by Cofactor Expansion
- Determinants by Row Reduction
- Properties of the Determinant
- The Adjoint and the Inverse (theory)
- Cramer's Rule
- Determinants, Invertibility and Solution Type (theory)

Determinants by Cofactor Expansion

1. Let

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 0 & 4 & 1 \\ 5 & -1 & 2 \end{bmatrix}.$$

a) Compute $\det(A)$ by cofactor expansion along the first row, writing out the three cofactors with their signs.

b) Recompute $\det(A)$ by expanding along the first column, and state what the agreement of the two values illustrates.

Determinants by Row Reduction

2. Evaluate $\det(A)$ for

$$A = \begin{bmatrix} 0 & 1 & 3 & 1 \\ 1 & 2 & -1 & 3 \\ 2 & 5 & 0 & 4 \\ 6 & 14 & -4 & 16 \end{bmatrix}$$

by reducing A to upper triangular form. State, beside each operation, its effect on the determinant, and give the value.

Properties of the Determinant

3. A and B are 3×3 matrices with $\det(A) = 5$ and $\det(B) = -2$. Evaluate each of the following, naming the property used, or state that the information given is not enough.

a) $\det(A^T B)$

b) $\det(2A)$

c) $\det(A^{-1} B^2)$

d) $\det\big((AB)^{-1}\big)$

e) $\det(A + B)$

The Adjoint and the Inverse

4. For

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 2 & 4 & 5 \end{bmatrix},$$

compute $\det(A)$, build the matrix of cofactors, write down $\operatorname{adj}(A)$, and hence obtain A^{-1} from $A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$. Verify your inverse by computing one column of AA^{-1}.

Cramer's Rule

5. Solve

$$\begin{aligned} 2x + y - z &= 3 \\ x - y + 2z &= 1 \\ 3x + 2y + z &= 8 \end{aligned}$$

by Cramer's rule. Show the four determinants you use, and check your solution in the second equation.

Determinants, Invertibility and Solution Type

6. Let

$$A(k) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}.$$

a) Show that $\det\big(A(k)\big) = (k-1)(k-2)$, and state all k for which $A(k)$ is invertible.

b) Take a value of k making $\det\big(A(k)\big) = 0$. What does that tell you about the number of solutions of the homogeneous system $A(k)\mathbf{x} = \mathbf{0}$? And about $A(k)\mathbf{x} = \mathbf{b}$ for an arbitrary $\mathbf{b}$?

c) For $k = 1$, exhibit a nonzero solution of $A(1)\mathbf{x} = \mathbf{0}$.

Synthesis — drawing on several topics in this set

7. Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 6 \\ 7 \\ 11 \end{bmatrix}.$$

a) Find $\det(A)$ by row reduction, stating the effect of each operation.

b) State, with the reason, whether $A\mathbf{x} = \mathbf{b}$ has a unique solution.

c) Use Cramer's rule to find x alone.

d) Evaluate $\det(A^{-1})$, $\det(2A^T)$ and $\det(A^3)$.