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Linear Algebra

Name: ______________________________ Date: ______________

Concepts

- Adding and Subtracting Vectors
- Comparing Two Vectors
- Linear Algebra (theory)
- Linear Combination of Vectors
- Matrices
- Matrix Operations
- Multiplying Vectors by a Scalar and the Scalar Product (Dot Product)
- Proving Vector Statements (theory)
- Solving First- or Second-Degree System of Equations (Semi-Linear)
- Solving Problems Involving Vectors
- The Orthogonal Projection of a Vector
- The Properties of Vector Operations (theory)
- Transformation Matrices
- Vector Components
- Vectors

Adding and Subtracting Vectors

1. Let $\vec{u} = (5, -2)$ and $\vec{v} = (-3, 6)$.

a) Find $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$.

b) Give the exact value of $\|\vec{u} + \vec{v}\|$, then a decimal value to the nearest hundredth.

c) Using the Chasles relation, simplify $\overrightarrow{RS} + \overrightarrow{TR} - \overrightarrow{US}$ into a single vector.

Comparing Two Vectors

2. Consider $\vec{a} = (4, -6)$, $\vec{b} = (-6, -4)$, $\vec{c} = (-2, 3)$, $\vec{d} = (-4, 6)$ and $\vec{e} = (6, -9)$. Support every answer with a calculation.

a) Which vector is the opposite of $\vec{a}$?

b) Which vectors are collinear with $\vec{a}$ without being equal or opposite to it?

c) Which vector is orthogonal to $\vec{a}$?

d) For $P(1, 1)$ and $Q(5, -5)$, is $\overrightarrow{PQ}$ equal to $\vec{a}$?

Linear Algebra

3. For each expression below, state whether the result is a *scalar*, a *vector*, a *matrix* (give its dimension) or is *not defined*, and say why. Here $\vec{u}$ and $\vec{v}$ are vectors of the plane, A is a 2×3 matrix, B is a 2×3 matrix and C is a 3×4 matrix.

a) $\vec{u} \cdot \vec{v}$

b) $\|\vec{u}\| \, \vec{v}$

c) $\vec{u} + \|\vec{v}\|$

d) AB

e) AC

Linear Combination of Vectors

4. Let $\vec{u} = (2, 1)$ and $\vec{v} = (-1, 3)$.

a) Write $\vec{w} = (8, -3)$ as a linear combination $a\vec{u} + b\vec{v}$.

b) Do the same for $\vec{z} = (-6, 11)$.

c) Explain why *every* vector of the plane can be written as a linear combination of $\vec{u}$ and $\vec{v}$.

Matrices

5. Let

$$A = \begin{pmatrix} 3 & -1 & 5 \\ 0 & 2 & -4 \end{pmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} -2 \\ 7 \\ 0 \end{bmatrix}.$$

a) Give the dimension of A and the value of the element a_{23}.

b) Write A^T and give its dimension.

c) Name the type of each of B and D.

Matrix Operations

6. Let $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 \\ -2 & 5 \end{pmatrix}$. Compute:

a) $2A - B$

b) AB

Multiplying Vectors by a Scalar and the Scalar Product (Dot Product)

7. Let $\vec{u} = (6, -3)$ and $\vec{v} = (-2, 5)$.

a) Compute $2\vec{u} - 3\vec{v}$.

b) Compute $\vec{u} \cdot \vec{v}$.

c) Find the angle between $\vec{u}$ and $\vec{v}$, to the nearest degree.

Proving Vector Statements

8. In a triangle ABC, let M be the midpoint of $\overline{AB}$ and N the midpoint of $\overline{AC}$.

a) Using only the Chasles relation and the properties of vector operations, prove that $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$.

b) State the two geometric facts about the segments $\overline{MN}$ and $\overline{BC}$ that follow from this vector equality.

Solving First- or Second-Degree System of Equations (Semi-Linear)

9. Solve the system algebraically and interpret the result geometrically.

$$\begin{cases} y = 2x + 1 \\ y = x^2 - 3x + 7 \end{cases}$$

Solving Problems Involving Vectors

10. Two tugboats pull a barge. Tug A pulls with a force of 4000 N directed 40° above the eastward axis; tug B pulls with 3000 N directed 25° below that same axis.

a) Write each force in components (nearest newton).

b) Find the norm of the resultant force, to the nearest newton, and its direction relative to the eastward axis, to the nearest tenth of a degree.

The Orthogonal Projection of a Vector

11. Let $\vec{u} = (7, 1)$ and $\vec{v} = (3, 4)$.

a) Find the orthogonal projection of $\vec{u}$ onto $\vec{v}$ and give its norm.

b) Compute $\vec{u} - \vec{u}_{\vec{v}}$ and verify that it is orthogonal to $\vec{v}$.

The Properties of Vector Operations

12. For each statement, decide whether it is true for all vectors $\vec{u}, \vec{v}, \vec{w}$ of the plane and all $k \in \mathbb{R}$. If it is true, name the property; if it is false, give an explicit counterexample.

a) $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$

b) $\|k\vec{u}\| = k\,\|\vec{u}\|$

c) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

d) $(\vec{u} \cdot \vec{v})\,\vec{w} = \vec{u}\,(\vec{v} \cdot \vec{w})$

Transformation Matrices

13. Points of the plane are written as column matrices $\begin{bmatrix} x \\ y \end{bmatrix}$ and transformed by multiplying on the left by a 2×2 matrix. Consider the triangle $A(2,1)$, $B(5,1)$, $C(5,3)$.

a) Write the matrix R of the rotation of $90°$ counterclockwise about the origin, and find the images A', B', C'.

b) Write the matrix H of the dilation centred at the origin with ratio 3, and find the image of A.

Vector Components

14.

a) A vector $\vec{u}$ has norm 12 and its direction makes an angle of $210°$ with the positive x-axis. Give its components, exactly and then to the nearest hundredth.

b) Give the norm of $\vec{v} = (-5, 12)$ and its direction, to the nearest tenth of a degree.

Vectors

15. Let $P(-3, 2)$ and $Q(5, -4)$.

a) Give the components of $\overrightarrow{PQ}$ and its norm.

b) Give the unit vector having the same direction and sense as $\overrightarrow{PQ}$.

c) Find the coordinates of the point R such that $\overrightarrow{QR} = \overrightarrow{PQ}$.

Synthesis — drawing on several sheets in this topic

16. A logo is built from the triangle $O(0,0)$, $P(4,0)$, $Q(4,3)$. It is first rotated 90° counterclockwise about the origin, then enlarged by a dilation centred at the origin with ratio 2.

a) Write the single matrix M that performs both transformations, and state whether reversing their order would change M.

b) Find the image Q' of Q.

c) Using the dot product and norms, verify that $\|\overrightarrow{OQ'}\| = 2\,\|\overrightarrow{OQ}\|$ and that $\overrightarrow{OQ} \perp \overrightarrow{OQ'}$.