University Business Mathematics · Practice Worksheet · tutorinmontreal.ca

Matrices and Input-Output Models

Name: ______________________________ Date: ______________

Concepts

- Modelling a Problem as a Linear System (theory)
- Augmented Matrices and Row Operations
- Gauss-Jordan Elimination for a Unique Solution
- Systems with No Solution or Infinitely Many Solutions
- Matrix Sums, Scalar Multiples and Products
- The Inverse Matrix and Matrix Equations
- Leontief Input-Output Analysis (theory)

Modelling a Problem as a Linear System

1. A bike shop assembles three models: city bikes, hybrids and cargo bikes. Each city bike needs 3 hours of frame assembly, 1 hour of wheel building and 1 hour of testing; each hybrid needs 4, 2 and 1 hours; each cargo bike needs 6, 2 and 3 hours. Next week the shop has 120 hours of frame assembly, 46 hours of wheel building and 45 hours of testing, and the owner wants every one of those hours used.

a) Define your variables and write the system of equations, with the variables in the same order in every equation.

b) The owner's first idea is 16 city bikes, 12 hybrids and 4 cargo bikes; her assistant suggests 12 city bikes, 9 hybrids and 8 cargo bikes. Check each plan against the system and say which one uses all the hours.

Augmented Matrices and Row Operations

2. Consider the system

$$y + 2x - z = 5, \qquad 3z - x = 4, \qquad x + y + z = 6.$$

a) Write its augmented matrix, with the variables in the order x, y, z.

b) Apply, in this order, $R_1 \leftrightarrow R_3$, then $R_2 \to R_2 + R_1$, then $R_3 \to R_3 - 2R_1$, and write the matrix after the three operations.

c) Apply $R_3 \to R_3 + R_2$, then finish by back-substitution to find x, y and z.

3. For each pair, name the single row operation that turns the first augmented matrix into the second, written in the form $R_2 \to R_2 - 3R_1$, $R_1 \to \frac{1}{2}R_1$ or $R_1 \leftrightarrow R_2$. If no single row operation does it, say so and explain why.

a) $\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & 5 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -1 & -10 \end{array}\right]$

b) $\left[\begin{array}{cc|c} 4 & 8 & 12 \\ 1 & 5 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 5 & 2 \end{array}\right]$

c) $\left[\begin{array}{cc|c} 0 & 2 & 6 \\ 3 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 3 & 1 & 5 \\ 0 & 2 & 6 \end{array}\right]$

d) $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 11 \end{array}\right]$

Gauss-Jordan Elimination for a Unique Solution

4. Solve by Gauss-Jordan elimination, carrying the augmented matrix all the way to reduced row echelon form and naming each row operation:

$$x + 3y + z = 10, \qquad 2x + 5y + 4z = 24, \qquad x + 2y + 4z = 17.$$

5. A concert hall sold 1 500 tickets for one show in three sections: floor seats at \$80, lower balcony at \$60 and upper balcony at \$35. Ticket revenue was \$84 000, and the upper balcony sold twice as many tickets as the floor. Set up a system and solve it by Gauss-Jordan elimination to find how many tickets of each kind were sold.

Systems with No Solution or Infinitely Many Solutions

6. Solve each system by Gauss-Jordan elimination. Say whether it has no solution, exactly one solution or infinitely many solutions, and if there are infinitely many, write the solution set using a parameter t.

a) $x + y + 2z = 5, \qquad 3x + 4y + 5z = 18, \qquad 2x + 3y + 3z = 10$

b) $x - y + 3z = 2, \qquad 2x - y + 4z = 7, \qquad x + z = 5$

7. An events company must staff a banquet with 20 workers: chefs paid \$30 an hour, servers paid \$20 an hour and dishwashers paid \$18 an hour. The client's budget fixes the total hourly wage bill at \$440.

a) Set up the system and solve it by Gauss-Jordan elimination, writing the general solution with the number of dishwashers d as the parameter.

b) Find the staffing if exactly 5 dishwashers are hired.

c) List every staffing that is possible in whole numbers of workers, with at least one worker of each kind.

Matrix Sums, Scalar Multiples and Products

8. A bike shop has two stores (rows: downtown, west end) and sells three models (columns: city, hybrid, cargo). For one month, S is the stock at the start, D the deliveries received and T the bikes sold:

$$S = \begin{bmatrix} 12 & 8 & 5 \\ 9 & 6 & 4 \end{bmatrix}, \qquad D = \begin{bmatrix} 10 & 6 & 4 \\ 8 & 10 & 2 \end{bmatrix}, \qquad T = \begin{bmatrix} 15 & 9 & 6 \\ 11 & 12 & 3 \end{bmatrix}.$$

a) Compute the end-of-month stock $E = S + D - T$ and say what the entry in row 2, column 3 means.

b) Next month the owner will order 1.5 times this month's deliveries. Compute $1.5D$.

9. A food truck sells poutine, sandwiches and smoothies at \$12, \$9 and \$6; each costs it \$5, \$4 and \$2 to make. Write these as the row matrices $P = \begin{bmatrix} 12 & 9 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & 4 & 2 \end{bmatrix}$. Last weekend's sales were

$$Q = \begin{bmatrix} 40 & 55 \\ 30 & 20 \\ 50 & 70 \end{bmatrix} \qquad \text{(rows: poutine, sandwiches, smoothies; columns: Saturday, Sunday).}$$

a) Compute PQ and CQ, and say what each entry means.

b) Compute $(P - C)Q$ and check that it equals $PQ - CQ$. What does it measure?

The Inverse Matrix and Matrix Equations

10. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

a) Find A^{-1} by row-reducing the augmented block $[\, A \mid I_3 \,]$ to $[\, I_3 \mid A^{-1} \,]$, naming each row operation.

b) Check your answer by computing AA^{-1}.

11. A sign shop makes two products: banners and vinyl decals. Each banner needs 2 hours of design and 1 hour of printing; each decal order needs 3 hours of design and 2 hours of printing. With x banners and y decal orders, the hours used are AX, where

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

For a 2×2 matrix, if $ad - bc \neq 0$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

a) Find A^{-1}.

b) In week 1 the shop has 36 design hours and 22 printing hours; in week 2, 50 and 30. Use A^{-1} to find the production that uses all the hours in each week.

Leontief Input-Output Analysis

12. A regional economy has two sectors, farming and food processing. Its technology matrix is

$$M = \begin{bmatrix} 0.1 & 0.4 \\ 0.3 & 0.2 \end{bmatrix} \qquad \text{(rows and columns: farming, food processing)},$$

where column j lists the dollar value of each sector's output used up in producing \$1 of sector j's output. If X is the total output and D the final (outside) demand, both in millions of dollars, then $X = MX + D$, that is $(I - M)X = D$.

a) Say in one sentence what the entry 0.3 means.

b) Find the total output of each sector needed to meet a final demand of $D = \begin{bmatrix} 30 \\ 60 \end{bmatrix}$.

Synthesis — drawing on several topics in this unit

13. A bakery makes batches of bread, croissants and cookies. The flour, butter and sugar (in kg) that one batch of each uses are the columns of

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \text{(rows: flour, butter, sugar; columns: bread, croissants, cookies).}$$

Its stock is $B = \begin{bmatrix} 64 \\ 34 \\ 26 \end{bmatrix}$ kg of flour, butter and sugar.

a) The manager plans 10 batches of bread, 8 of croissants and 5 of cookies. Compute the ingredients this uses and the stock left over.

b) Find, by Gauss-Jordan elimination, the numbers of batches that would use the whole stock exactly.

Want the harder problems and the worked solutions?

The *Challenge Problems* set and the *Answer Keys* for Matrices and Input-Output Models — with the reasoning written out in full — are available at **tutorinmontreal.ca**.