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Differentiation Techniques and Elasticity

Name: ____ Date: __________________

Concepts

- Derivatives of Exponential and Logarithmic Functions
- The Product and Quotient Rules
- The Chain Rule with Powers, Exponentials and Logarithms
- Implicit Differentiation (theory)
- Related Rates in Business Settings (theory)
- Relative Rate of Change and Elasticity of Demand

Derivatives of Exponential and Logarithmic Functions

1. Differentiate each function, for $x > 0$. Use only the derivatives of e^x, b^x, $\ln x$ and $\log_b x$ and the sum and constant-multiple rules; rewrite first where that helps.

a) $f(x) = 5e^x - 8\ln x + 2x^3$, and then evaluate $f'(1)$ exactly.

b) $g(x) = 3 \cdot 2^x + \log_{10} x$

c) $h(x) = \ln\left(\dfrac{x^4}{e^2}\right)$

2. A courier company buys a delivery van for \$45 000. Its book value t years later is modelled by

$$V(t) = 45\,000\,(0.8)^t \text{ dollars}, \qquad t \geq 0.$$

a) Find $V'(t)$.

b) Find $V'(0)$ and $V'(3)$, exactly and to the nearest dollar, and state their units. Use $\ln 0.8 \approx -0.22314$.

c) In one sentence, say what the comparison of $V'(0)$ and $V'(3)$ tells the owner.

The Product and Quotient Rules

3. Differentiate and simplify, for $x > 0$.

a) $f(x) = x^3 e^x$ (factor your answer)

b) $g(x) = (3x - 2)\ln x$

c) $h(x) = \dfrac{e^x}{x+2}$ (factor e^x out of the numerator)

d) $k(x) = \dfrac{\ln x}{x^2}$

4. A yoga studio sells monthly memberships. Its monthly revenue is $R = p\,q$, where p is the membership price in dollars and q the number of members, both functions of the time t in months. Right now the price is $60 and is being raised by $1.50 a month, and the studio has 450 members, a number that is falling by 6 members a month.

a) Is monthly revenue rising or falling right now, and at what rate?

b) With the price still rising by $1.50 a month, how fast would membership have to be falling right now for revenue to be momentarily neither rising nor falling?

The Chain Rule with Powers, Exponentials and Logarithms

5. Differentiate each function, simplifying where a common factor allows it.

a) $f(x) = \left(x^2 - 3x + 5\right)^3$

b) $g(x) = 3e^{x^2-4x}$

c) $h(x) = \ln\left(x^2 + 2x + 10\right)$

d) $k(x) = \sqrt{4x+9}$, for $x > -\frac{9}{4}$

e) $m(x) = 2^{3x+1}$

6. A maker of board games finds that to sell x copies a week it must set the price

$$p = D(x) = 80e^{-x/400} \text{ dollars}, \qquad x \geq 0.$$

a) Find $D'(400)$ exactly and to four decimal places, with units, using $e^{-1} \approx 0.3679$.

b) Write the weekly revenue $R(x)$ and show that $R'(x) = 80e^{-x/400}\left(1 - \dfrac{x}{400}\right)$.

c) Find $R'(200)$ and $R'(600)$ exactly and to the cent, using $e^{-1/2} \approx 0.60653$ and $e^{-3/2} \approx 0.22313$, and say what the sign of each tells the maker.

Implicit Differentiation

7. The weekly demand x for a portable speaker and its price p, in dollars, satisfy

$$x^2 + 25p^2 = 62\,500, \qquad x \geq 0,\ p \geq 0.$$

a) Check that $(x, p) = (200, 30)$ satisfies the relation.

b) Differentiate implicitly with respect to x to find $\dfrac{dp}{dx}$ in terms of x and p.

c) Evaluate $\dfrac{dp}{dx}$ at $(200, 30)$ and say what it means, with units.

Related Rates in Business Settings

8. A skate shop sells x pairs of skates a month at a price of p dollars a pair, where

$$x = 1200 - 0.2p^2.$$

The price is \$50 a pair and is being raised by \$2 a month.

a) How fast is the number of pairs sold changing?

b) How fast is the monthly revenue $R = xp$ changing?

Relative Rate of Change and Elasticity of Demand

9. The relative rate of change of a quantity $f(t)$ is $\dfrac{f'(t)}{f(t)}$; multiplied by 100 it is the percentage rate of change.

a) A franchise's annual revenue is $R(t) = 3.2e^{0.06t}$ million dollars, t years from now. Find its percentage rate of change.

b) A café's monthly customer count is $N(t) = 2000 + 80t$, t months from now. Find its percentage rate of change at $t = 5$ and at $t = 20$.

c) The café gains the same 80 customers every month. Explain why its percentage rate of growth is nevertheless falling.

10. A climbing gym sells x day passes a week at a price of p dollars, where

$$x = f(p) = 900 - p^2, \qquad 0 < p < 30.$$

The elasticity of demand is $E(p) = -\dfrac{p\,f'(p)}{f(p)}$; demand is elastic where $E(p) > 1$ and inelastic where $E(p) < 1$.

a) Find $E(p)$.

b) Find $E(10)$ and $E(20)$, and classify demand at each price.

c) At \$20, estimate the percentage change in demand if the price rises by 2%, and say whether weekly revenue rises or falls.

d) Find the price at which demand has unit elasticity, exactly and to the cent, using $\sqrt{3} \approx 1.7321$.

Synthesis — drawing on several topics in this unit

11. A meal-kit company sells x boxes a week at a price of p dollars a box, where

$$x = f(p) = \frac{3\,600\,000}{(p+20)^2}, \qquad p > 0.$$

The elasticity of demand is $E(p) = -\dfrac{p\,f'(p)}{f(p)}$.

a) Find $f'(p)$.

b) Show that $E(p) = \dfrac{2p}{p+20}$.

c) Classify demand at \$10 and at \$40, and say for each whether a small price increase raises or lowers weekly revenue.

d) Find the price at which demand has unit elasticity, and the number of boxes sold there.