University Calculus III · Practice Worksheet · tutorinmontreal.ca

Directional Derivatives and the Gradient

Name: ______________________________ Date: ______________

Concepts

- The Gradient Vector
- Directional Derivatives
- The Direction of Maximum Rate of Change
- The Gradient and Level Curves (theory)
- Tangent Planes and Normal Lines to a Level Surface

The Gradient Vector

1. Find the gradient of each function, then evaluate it at the given point.

a) $f(x, y) = y \ln x + xy^2$, for $x > 0$, at the point $(1, 2)$.

b) $g(x, y, z) = xz^2 + y^2 z - 3xy$ at the point $(1, -1, 2)$.

2. Let $f(x, y) = x^2 - xy + 2y^2$.

a) Find the point at which $\nabla f = (4, 5)$.

b) Describe the set of all points at which ∇f points in the direction of the positive y-axis, that is, $\nabla f = (0, c)$ with $c > 0$.

Directional Derivatives

3. Find the directional derivative of $f(x, y) = x^2 - 3xy + y^3$ at the point $(1, 2)$ in the direction of $\vec{v} = (5, -12)$. State whether f is increasing or decreasing in that direction.

4. The concentration of a dye in a tank, in mg/L, is $C(x, y, z) = x^2y - xz + yz^2$, with x, y, z in centimetres.

a) Find the rate of change of the concentration at the point $P(2, 1, -1)$ in the direction from P towards the point $Q(4, -1, 0)$. Include units.

b) A probe passes through P heading towards Q at 3 cm/s. At what rate, in mg/L per second, is the concentration it reads changing at that instant?

The Direction of Maximum Rate of Change

5. Let $f(x, y) = \sqrt{x^2 + 2y}$, defined for $x^2 + 2y > 0$. At the point $(1, 4)$, find the unit direction in which f increases most rapidly and the maximum rate of increase. Then state the unit direction in which f decreases most rapidly, and the value of the directional derivative of f in that direction (a signed number).

6. The temperature at the point (x, y) of a heated plate is $T(x, y) = 50 - x^2 - y^2 - 4y$, in °C, with x and y in centimetres. An ant is at the point $(3, -2)$.

a) In which unit direction should the ant set off to warm up as fast as possible, and what is that rate?

b) The ant prefers to warm up at exactly 3 °C/cm. Find every unit direction that achieves this.

c) Explain why no direction lets the ant warm up at 7 °C/cm.

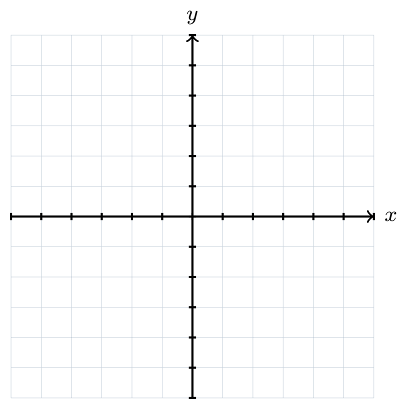
The Gradient and Level Curves

7. Let $f(x,y) = \frac{1}{2}x^2 - y$. On the grid below, the axes cross at the centre and each square is one unit, so the grid shows $-6 \le x \le 6$ and $-6 \le y \le 6$.

a) Write the equation of the level curve of f that passes through the point $(2,1)$, and sketch the part of it that fits on the grid.

b) Compute $\nabla f(2,1)$ and draw it on the grid with its tail at $(2,1)$.

c) Use the gradient to find the equation of the tangent line to the level curve at $(2,1)$, and draw the line.



Tangent Planes and Normal Lines to a Level Surface

8. Find an equation of the tangent plane, and parametric equations of the normal line, to the surface $x^2 + 2y^2 - z^2 = 5$ at the point $(2,1,1)$.

9. Find the point on the paraboloid $z = x^2 + 3y^2$ at which the tangent plane is parallel to the plane $4x - 6y - z = 20$, and give an equation of that tangent plane.

Synthesis — drawing on several topics in this unit

10. The elevation of a hillside, in metres, is $h(x, y) = 500 - \dfrac{x^2}{400} - \dfrac{y^2}{100}$, where x is the distance east and y the distance north of a marker, both in metres. A hiker stands at the point above $(80, 30)$.

a) Find the hiker's elevation and $\nabla h(80, 30)$.

b) The hiker sets off towards the north-east, the direction of $(1, 1)$. Is she climbing or descending at first, and at what slope (metres of elevation per metre travelled horizontally)?

c) In which unit direction is the hill steepest uphill, and what is the slope in that direction?

d) Give the two unit directions in which she could set off without changing elevation at first, and the equation of the tangent line at $(80, 30)$ to the contour line she is standing on.