University Calculus III · Practice Worksheet · tutorinmontreal.ca

Optimization in Several Variables

Name: ____ Date: __________________

Concepts

- Critical Points of a Function of Two Variables
- Second-Order Taylor Approximation in Two Variables (theory)
- The Second Derivative Test
- Absolute Extrema on a Closed Bounded Region
- Lagrange Multipliers with One Constraint
- Applied Optimization Problems in Several Variables (theory)

Critical Points of a Function of Two Variables

1. Find every critical point of

$$f(x,y) = 2x^2 + y^2 - 2xy - 6x + 2y,$$

and give the value of f there. Do not classify the point.

2. Find every critical point of

$$f(x,y) = x^2y - 4xy + y^2.$$

Show how each one arises. Do not classify them.

Second-Order Taylor Approximation in Two Variables

3. Let $f(x, y) = \dfrac{x^2}{y}$ for $y > 0$.

a) Find the second-order Taylor polynomial $T_2(x, y)$ of f about the point $(2, 1)$.

b) Use it to approximate $f(2.1,\ 0.9)$, and compare with the linear approximation and with the exact value.

The Second Derivative Test

4. Find and classify every critical point of

$$f(x, y) = 2x^3 - 6xy + 3y^2.$$

5. Find and classify every critical point of

$$f(x,y) = \left(x^2 + y^2 - 3\right) e^x.$$

Absolute Extrema on a Closed Bounded Region

6. A thin triangular metal plate occupies the closed region with vertices $(0,0)$, $(3,0)$ and $(0,3)$, with distances in centimetres. Its temperature, in degrees Celsius, is

$$T(x,y) = x^2 - 2xy + 2y^2 - 2y + 5.$$

Find the hottest and the coldest points of the plate and the temperature at each.

7. Find the absolute maximum and the absolute minimum of

$$f(x, y) = x^2 + 2y^2 - 2x$$

on the closed disc $x^2 + y^2 \leq 4$, and every point at which each is attained.

Lagrange Multipliers with One Constraint

8. Use a Lagrange multiplier to find the maximum and the minimum of

$$f(x, y) = 4x + 5y \qquad \text{subject to} \qquad x^2 + xy + y^2 = 7.$$

The constraint curve is an ellipse.

9. Use a Lagrange multiplier to find the maximum and the minimum of

$$f(x,y) = xy^2 \qquad \text{subject to} \qquad x^2 + y^2 = 3.$$

List *every* point the method produces, including any that turn out to give neither extreme.

Applied Optimization Problems in Several Variables

10. Three monitoring stations stand at $A(0,0)$, $B(8,0)$ and $C(2,6)$, with coordinates in kilometres. A radio repeater is to be placed at a point $P(x,y)$. The transmission energy it uses is proportional to

$$S = |PA|^2 + |PB|^2 + 2\,|PC|^2,$$

the last term being doubled because station C reports twice as often as the others. Find the position of P that minimizes S, justify that it is a minimum, and give the minimum value of S.

Synthesis — drawing on several topics in this unit

11. Let $f(x, y) = 2x^3 - 24x + 9y^2$.

a) Find and classify the critical points of f.

b) Find the absolute maximum and the absolute minimum of f on the closed disc $x^2 + y^2 \leq 9$, and every point at which each occurs.

c) One of the two critical points plays no part in the answer to (b). Which one, and why could that have been predicted from (a)?