University Calculus III · Practice Worksheet · tutorinmontreal.ca

Double Integrals

Name: ____ Date: __________________

Concepts

- Double Integrals over Rectangles and Iterated Integrals
- Double Integrals over General Regions
- Reversing the Order of Integration
- Polar Coordinates and Polar Curves (theory)
- Double Integrals in Polar Coordinates
- Area and Volume by Double Integration (theory)
- Mass and Centre of Mass of a Lamina (theory)

Double Integrals over Rectangles and Iterated Integrals

1. Let $R = [1,3] \times [0,2]$, that is $1 \le x \le 3$ and $0 \le y \le 2$, and let $f(x,y) = 4xy + 3y^2$.

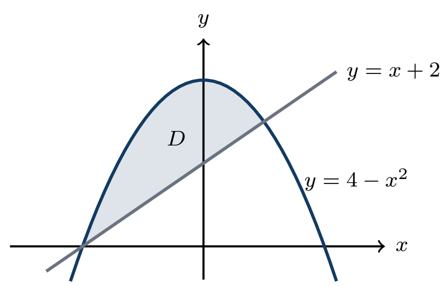
a) Evaluate $\iint_R f(x,y)\,dA$ as the iterated integral $\int_1^3 \int_0^2 f(x,y)\,dy\,dx$.

b) Evaluate it again in the order $dx\,dy$, and state the theorem that says the two values had to agree.

2. Evaluate
$$\iint_R \frac{x}{(1+xy)^2}\,dA, \qquad R = [0,2]\times[0,1].$$
One order of integration is much shorter than the other. Say which you chose and why.

Double Integrals over General Regions

3. Let D be the region enclosed by the parabola $y = 4 - x^2$ and the line $y = x + 2$ (shaded below).



a) Find the two points where the curves meet, and describe D by inequalities of the form $a \le x \le b$, $g_1(x) \le y \le g_2(x)$.

b) Evaluate $\displaystyle\iint_D x\,dA$.

4. Let D be the triangle with vertices $(0,0)$, $(3,3)$ and $(6,0)$.

a) Describe D by inequalities of the form $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, and explain in one sentence why this description is more convenient here than one using vertical strips.

b) Evaluate $\displaystyle\iint_D xy\,dA$.

Reversing the Order of Integration

5. Consider $\displaystyle\int_0^3 \int_{x^2}^{3x} f(x,y)\,dy\,dx$.

a) Sketch the region of integration and label its boundary curves and corner points.

b) Rewrite the integral in the order $dx\,dy$.

c) Check your limits by evaluating both iterated integrals for $f(x,y) = x$.

6. Evaluate

$$\int_0^2 \int_{2y}^4 e^{-x^2/4}\, dx\, dy.$$

The function $e^{-x^2/4}$ has no antiderivative that can be written with elementary functions, so begin by reversing the order of integration.

Polar Coordinates and Polar Curves

7. Polar coordinates (r, θ) are related to Cartesian ones by $x = r\cos\theta$, $y = r\sin\theta$, with $r \geq 0$.

a) Find the polar equation of the circle $x^2 + y^2 = 2y$, and give an interval of θ, inside $0 \leq \theta \leq 2\pi$, over which the circle is traced exactly once.

b) Find the polar equation of the vertical line $x = 2$.

c) Let D be the part of the disc $x^2 + y^2 \leq 16$ lying to the right of the line $x = 2$. Describe D by inequalities of the form $\alpha \leq \theta \leq \beta$, $r_1(\theta) \leq r \leq r_2(\theta)$.

Double Integrals in Polar Coordinates

8. Let D be the part of the ring $1 \le x^2 + y^2 \le 4$ that lies in the first quadrant ($x \ge 0$, $y \ge 0$). Describe D in polar coordinates and evaluate $\iint_D xy\, dA$.

9. Sketch the region of integration of

$$\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} \frac{1}{1 + x^2 + y^2}\, dy\, dx,$$

convert the integral to polar coordinates, and evaluate it.

Area and Volume by Double Integration

10. Let D be the triangle in the xy-plane with vertices $(0, 0)$, $(2, 0)$ and $(2, 2)$. A solid lies above D and below the surface $z = 4 - x^2$.

a) Write the area of D as a double integral and evaluate it.

b) Check that $z \geq 0$ on D, then write the volume of the solid as a double integral and evaluate it.

Mass and Centre of Mass of a Lamina

11. A thin plate (a lamina) occupies the triangle D with vertices $(0, 0)$, $(1, 0)$ and $(1, 2)$, where x and y are in centimetres. Its density at (x, y) is $\delta(x, y) = 6x + 6y$ grams per square centimetre. Find the mass of the plate and the coordinates $(\bar{x}, \bar{y})$ of its centre of mass.

Synthesis — drawing on several topics in this unit

12. A lamina occupies the region D in the first quadrant that lies inside the circle $x^2 + y^2 = 4$ and between the x-axis and the line $y = x$. Lengths are in centimetres and the density is $\delta(x, y) = y$ grams per square centimetre.

a) Describe D in polar coordinates, and also by inequalities of the form $c \le y \le d$, $h_1(y) \le x \le h_2(y)$.

b) Find the mass of the lamina using polar coordinates.

c) Set up the mass as a Cartesian iterated integral using your second description, and evaluate it to confirm b).