University Calculus III · Practice Worksheet · tutorinmontreal.ca

Triple Integrals and Change of Variables

Name: ____________________________________ Date: ________________

Concepts

- Triple Integrals in Cartesian Coordinates
- Setting Up the Limits for a Solid Region
- Triple Integrals in Cylindrical Coordinates
- Triple Integrals in Spherical Coordinates
- Volume, Mass and Centre of Mass of a Solid (theory)
- The Jacobian and Change of Variables

Triple Integrals in Cartesian Coordinates

1. Let B be the rectangular box $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$. Evaluate

$$\iiint_B (2x + yz)\, dV.$$

2. Consider the iterated integral

$$\int_0^1 \int_0^z \int_0^{y+z} 6x \, dx\, dy\, dz.$$

a) Describe the solid of integration by a system of inequalities.

b) Evaluate the integral.

Setting Up the Limits for a Solid Region

3. Let E be the solid in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) lying under the plane $2x + y + 2z = 4$.

a) Write $\iiint_E f(x, y, z)\, dV$ as an iterated integral in the order $dz\, dy\, dx$.

b) Use your limits to evaluate $\iiint_E x\, dV$.

4. The solid E is bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $y = 0$, $z = 0$ and $y + z = 4$; that is, E is the set of points with $0 \le z \le 4 - x^2$ and $0 \le y \le 4 - z$.

a) Write the volume of E as an iterated integral in the order $dy\,dz\,dx$.

b) Write it again in the order $dx\,dz\,dy$.

c) Evaluate whichever of the two is easier.

Triple Integrals in Cylindrical Coordinates

5. The solid E lies inside the cylinder $x^2 + y^2 = 9$, above the plane $z = 0$ and below the cone $z = \sqrt{x^2 + y^2}$. Use cylindrical coordinates (r, θ, z) to evaluate

$$\iiint_E z\,dV.$$

6. Convert the integral

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} y\,dz\,dy\,dx$$

to cylindrical coordinates (r,θ,z), stating the range of each variable, and evaluate it.

Triple Integrals in Spherical Coordinates

7. Use spherical coordinates (ρ,θ,ϕ) in which ϕ is measured from the positive z-axis $(0 \le \phi \le \pi)$ and θ is the same angle as in cylindrical coordinates, so that $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$ and $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

Let E be the part of the shell $1 \le x^2+y^2+z^2 \le 4$ that lies in the first octant ($x \ge 0$, $y \ge 0$, $z \ge 0$). Evaluate

$$\iiint_E (x^2+y^2+z^2)\,dV.$$

8. Use spherical coordinates (ρ, θ, ϕ) in which ϕ is measured from the positive z-axis $(0 \leq \phi \leq \pi)$ and θ is the same angle as in cylindrical coordinates, so that $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$ and $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

The solid E lies inside the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and *below* the cone $z = \sqrt{\frac{1}{3}(x^2 + y^2)}$. Find the range of each spherical variable on E, and evaluate

$$\iiint_E z \, dV.$$

Volume, Mass and Centre of Mass of a Solid

9. A solid cone occupies the region $\sqrt{x^2+y^2} \le z \le 2$, with lengths in metres. Its density at each point is numerically equal to the height of the point: $\delta(x,y,z) = z$ kg/m^3. Find the mass of the cone and the coordinates of its centre of mass.

The Jacobian and Change of Variables

10. Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of each transformation.

a) $x = 3u - v,\ \ y = u + 2v.$

b) $x = \dfrac{u}{v},\ \ y = uv$, for $u > 0$, $v > 0$. Also give its value at $(u,v) = (6,2)$.

11. Let R be the parallelogram in the xy-plane bounded by the lines $x - y = 0$, $x - y = 2$, $x + 2y = 0$ and $x + 2y = 3$. Use the change of variables $u = x - y$, $v = x + 2y$ to evaluate

$$\iint_R (x-y)^2(x+2y)\,dA.$$

Synthesis — drawing on several topics in this unit

12. The solid E is enclosed between the two paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$, with lengths in centimetres.

a) Write the volume of E as an iterated integral in Cartesian coordinates, in the order $dz\,dy\,dx$. Do not evaluate it.

b) Convert to cylindrical coordinates and find the volume.

c) The solid has density $\delta(x, y, z) = x^2 + y^2$ g/cm^3. Find its mass.

d) State the centre of mass of the solid in part c), with a reason and without further integration.