University Calculus III · Practice Worksheet · tutorinmontreal.ca

Vector Fields and Line Integrals

Name: ____ Date: __________________

Concepts

- Vector Fields
- Line Integrals of Scalar Functions (theory)
- Line Integrals of Vector Fields and Work
- Conservative Fields and Potential Functions
- The Fundamental Theorem for Line Integrals
- Green's Theorem
- Curl and Divergence (theory)

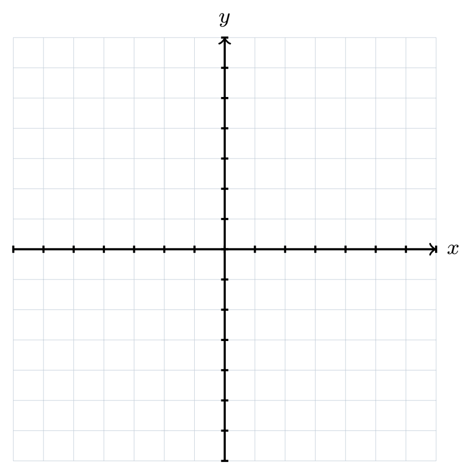
Vector Fields

1. Let $\vec{F}(x,y) = \left(\frac{1}{2}y,\ -\frac{1}{2}x\right)$.

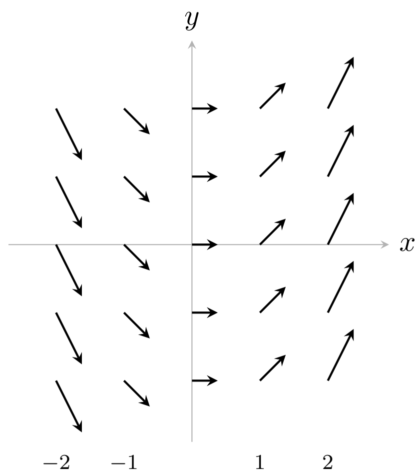
a) Compute $\vec{F}$ at the six points $(4,0)$, $(0,4)$, $(-4,0)$, $(0,-4)$, $(4,4)$ and $(-4,4)$, and draw each vector on the grid with its tail at the point where it was computed. Each square of the grid is one unit, and the origin is where the axes cross.

b) Let $\vec{r} = (x,y)$ be the position vector of a point. Show that $\vec{F}(x,y)\cdot\vec{r} = 0$ at every point, and express $\|\vec{F}(x,y)\|$ in terms of $\|\vec{r}\|$.

c) Use (b) to describe the whole field in one sentence.



2. The plot shows a vector field $\vec{F}$ on the square $-2 \le x \le 2$, $-2 \le y \le 2$. Three features of it can be read from the plot: (i) along any vertical line all the arrows are identical; (ii) on the y-axis the arrows are horizontal and point to the right; (iii) to the right of the y-axis the arrows point up and to the right, to the left of it they point down and to the right, and they get steeper the farther they are from the y-axis.



Which one of the following is $\vec{F}$? For each of the other three, name a feature among (i)–(iii) that it fails, with a point where the failure shows. (One feature is enough, even if the field fails more than one.)

$$\vec{A} = (x,\, 1), \qquad \vec{B} = (1,\, x), \qquad \vec{C} = (1,\, y), \qquad \vec{D} = (y,\, 1).$$

Line Integrals of Scalar Functions

3. A thin wire lies along the curve $\vec{r}(t) = \left(\frac{1}{3}t^3,\ t^2,\ 2t\right)$, $0 \le t \le 2$, with lengths in centimetres. Its linear density at the point (x, y, z) is $\delta(x, y, z) = z$ grams per centimetre.

a) Show that $\|\vec{r}\,'(t)\| = t^2 + 2$.

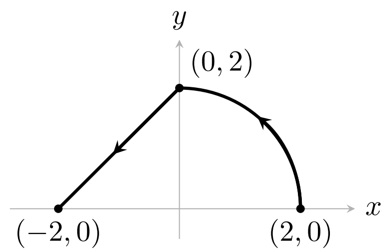
b) Find the mass of the wire, $\displaystyle\int_C \delta\, ds$.

c) Find the length of the wire and hence its average density.

Line Integrals of Vector Fields and Work

4. A force field $\vec{F}(x, y) = (xy,\ x - y)$, in newtons, acts on a particle that moves along the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$, with distances in metres. Find the work done by $\vec{F}$, $W = \displaystyle\int_C \vec{F} \cdot d\vec{r}$.

5. The curve C has two pieces: first the quarter of the circle $x^2 + y^2 = 4$ from $(2, 0)$ counterclockwise to $(0, 2)$, then the straight segment from $(0, 2)$ to $(-2, 0)$.



Evaluate $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x, y) = (-y,\ x)$.

Conservative Fields and Potential Functions

6. A plane vector field $\vec{F} = (P, Q)$ whose components have continuous partial derivatives on a region with no holes (a simply connected region, such as the whole plane) is conservative on that region exactly when $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ throughout it.

Let $\vec{F}(x, y) = \big(3x^2y - 2y^2,\ x^3 - 4xy + 6y\big)$ on $\mathbb{R}^2$.

a) Writing $\vec{F} = (P, Q)$, check that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, and say why this shows that $\vec{F}$ is conservative, checking each hypothesis of the test above.

b) Find a potential function f, that is, a function with $\nabla f = \vec{F}$.

7. A vector field $\vec{F} = (P, Q, R)$ whose components have continuous partial derivatives on all of $\mathbb{R}^3$ is conservative exactly when

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Let $\vec{F}(x, y, z) = \big(a\,xy + z^2,\ 3x^2,\ b\,xz\big)$, where a and b are constants.

a) Find the values of a and b for which $\vec{F}$ is conservative.

b) For those values, find a potential function $f(x, y, z)$.

The Fundamental Theorem for Line Integrals

8. Let $f(x, y) = x^2y^3 + y^2$ and $\vec{F} = \nabla f$.

a) Write out $\vec{F}(x, y)$.

b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve $\vec{r}(t) = \big(1 + t^2,\ \cos(\pi t)\big)$, $0 \le t \le 1$.

9. The force field $\vec{F}(x,y,z) = (yz,\ xz,\ xy + 2z)$, in newtons, has the potential function $f(x,y,z) = xyz + z^2$; distances are in metres. Let $A = (1,2,0)$, $B = (2,1,3)$ and $D = (0,5,1)$.

a) Verify that $\nabla f = \vec{F}$.

b) Find the work done by $\vec{F}$ on a particle that moves from A to B along any smooth curve.

c) The particle then continues from B to D. Find the work done on this second leg, and the total work from A to D.

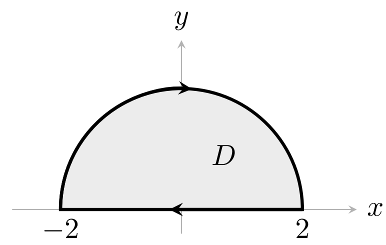
d) How much work does $\vec{F}$ do on a particle that leaves A and returns to A?

Green's Theorem

10. Let C be the boundary of the triangle with vertices $(0,0)$, $(3,0)$ and $(0,3)$, traversed counterclockwise. Use Green's theorem to evaluate

$$\oint_C y^2\,dx + 3xy\,dy.$$

11. Let D be the half-disk $x^2 + y^2 \leq 4$, $y \geq 0$, and let C be its boundary — the diameter from $(2, 0)$ to $(-2, 0)$ followed by the semicircle from $(-2, 0)$ over the top back to $(2, 0)$ — so that C is traversed **clockwise**.



Evaluate

$$\oint_C \left(\sqrt{1 + x^4} - y^3\right) dx + \left(x^3 + e^{y^2}\right) dy,$$

and say in one sentence why Green's theorem is the right tool here.

Curl and Divergence

12. Let $\vec{F}(x,y,z) = (x^2y,\ yz^2,\ xz - y)$.

a) Compute the divergence $\nabla \cdot \vec{F}$ and the curl $\nabla \times \vec{F}$.

b) Evaluate both at the point $(1, -1, 2)$.

c) Is $\vec{F}$ conservative? Give the reason.

Synthesis — drawing on several topics in this unit

13. A force field $\vec{F}(x, y) = (2xy,\ x^2 + 3x)$, in newtons, acts in the plane; distances are in metres.

a) Show that $\vec{F}$ is not conservative.

b) Use Green's theorem to find the work done by $\vec{F}$ on a particle that goes once counterclockwise round the circle $x^2 + y^2 = 4$.

c) Find the work done by $\vec{F}$ along the diameter from $(-2, 0)$ to $(2, 0)$.

d) Deduce, using Green's theorem on the upper half-disk, the work done along the upper semicircle from $(2, 0)$ counterclockwise to $(-2, 0)$.