University Calculus III · Practice Worksheet · tutorinmontreal.ca

Surface Integrals and Integral Theorems

Name: ______________________________ Date: ______________

Concepts

- Parametric Surfaces
- Surface Area (theory)
- Surface Integrals of Scalar Functions (theory)
- Flux Integrals of Vector Fields
- Stokes' Theorem
- The Divergence Theorem

Parametric Surfaces

1. A surface S is given by

$$\vec{r}(u,v) = (3\cos u,\ v,\ 3\sin u), \qquad 0 \le u \le 2\pi, \quad 0 \le v \le 4.$$

a) Eliminate the parameters to find a Cartesian equation satisfied by every point of S, and name the surface, including its axis and where it begins and ends.

b) Describe the grid curve obtained by holding $u = \frac{\pi}{2}$ fixed, and the grid curve obtained by holding $v = 1$ fixed.

2. Find an equation of the tangent plane to the parametric surface

$$\vec{r}(u,v) = (uv,\ u + v^2,\ u^2 - v)$$

at the point where $u = 1$ and $v = 2$.

Surface Area

3. Find the area of the part of the paraboloid $z = \frac{1}{2}\left(x^2 + y^2\right)$ that lies inside the cylinder $x^2 + y^2 = 3$.

Surface Integrals of Scalar Functions

4. A thin metal panel has the shape of the part of the plane $2x+2y+z=6$ that lies in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$), with lengths in metres. Its density at the point (x,y,z) is $\sigma(x,y,z) = z$ kg/m^2. Find the mass of the panel, $\iint_S \sigma \, dS$.

Flux Integrals of Vector Fields

5. Let S be the part of the paraboloid $z = 4 - x^2 - y^2$ that lies on or above the xy-plane, oriented upward. Evaluate the flux $\iint_S \vec{F} \cdot d\vec{S}$ of $\vec{F}(x,y,z) = (x,\, y,\, z)$ across S.

6. Let S be the cylinder $x^2 + y^2 = 4$, $0 \leq z \leq 3$ (the curved wall only, no top or bottom), oriented with the normal pointing away from the z-axis.

a) For the parametrization $\vec{r}(\theta, z) = (2\cos\theta,\, 2\sin\theta,\, z)$, compute $\vec{r}_\theta \times \vec{r}_z$ and decide whether it agrees with the required orientation.

b) Evaluate the flux of $\vec{F}(x, y, z) = (xz,\, yz,\, xy)$ across S.

Stokes' Theorem

7. Let C be the curve in which the plane $x + y + z = 5$ meets the cylinder $x^2 + y^2 = 9$, traversed counterclockwise when viewed from above, and let $\vec{F}(x, y, z) = (z,\, x^2,\, y)$. Use Stokes' theorem to evaluate $\oint_C \vec{F} \cdot d\vec{r}$.

8. Let S be the part of the paraboloid $z = 9 - x^2 - y^2$ that lies on or above the plane $z = 5$, oriented upward, and let $\vec{F}(x, y, z) = (-yz,\ xz,\ xyz)$. Use Stokes' theorem to evaluate $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$ as a line integral. State the boundary curve and its direction.

The Divergence Theorem

9. Let E be the box $0 \le x \le 2$, $0 \le y \le 1$, $0 \le z \le 3$, let S be its surface oriented outward, and let $\vec{F}(x, y, z) = (x^2,\ xy,\ 3z)$.

a) Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ using the divergence theorem.

b) Verify your answer by computing the flux through each of the six faces.

10. Let E be the solid cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 2$, and let S be its complete surface (wall, top and bottom) oriented outward. Evaluate the outward flux across S of

$$\vec{F}(x, y, z) = \left(xy^2 + \cos z,\ \ x^2y + e^{xz},\ \ z^2 + xy\right).$$

Synthesis — drawing on several topics in this unit

11. Let E be the solid bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$. Its boundary consists of the flat top S_1 (the disk $x^2 + y^2 \leq 4$ in the plane $z = 4$) and the curved bowl S_2 (the part of $z = x^2 + y^2$ with $z \leq 4$), both oriented outward from E. Let $\vec{F}(x, y, z) = (x,\ y,\ z^2)$.

a) State the direction (upward or downward) of the outward normal on S_1 and on S_2, and compute the flux of $\vec{F}$ across each piece.

b) Evaluate $\iiint_E \nabla \cdot \vec{F}\, dV$ in cylindrical coordinates and confirm that the divergence theorem holds.