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Mechanical Vibrations and Electric Circuits

Name: ______________________________ Date: ______________

Concepts

- Free Undamped Motion in Amplitude and Phase Form
- Free Damped Motion and the Three Damping Cases
- Driven Motion with Transient and Steady-State Terms
- Resonance (theory)
- Series Circuits with Resistance, Inductance and Capacitance

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Free Undamped Motion in Amplitude and Phase Form

1. A mass of 2 kg hangs from a spring with spring constant 50 N/m. There is no damping and no external force, so Newton's second law gives $m\dfrac{d^2x}{dt^2} + kx = 0$, where $x(t)$ is the displacement in metres from the equilibrium position (positive downward) and t is in seconds. At $t = 0$ the mass is 0.2 m below equilibrium and moving downward at $\sqrt{3}$ m/s.

a) Solve for $x(t)$.

b) Write the solution in amplitude-phase form $x = A\cos(\omega t - \delta)$ with $A > 0$ and $-\pi < \delta \leq \pi$. State the amplitude and the period.

c) Find the first time $t > 0$ at which the mass passes through equilibrium, and the greatest speed it ever reaches.

2. A mass of 0.5 kg stretches a spring by 0.2 m when it hangs at rest. Take $g = 9.8$ m/s^2. By Hooke's law the spring force is k times the stretch, and with no damping and no external force the displacement $x(t)$, in metres below equilibrium (negative above), satisfies $m\dfrac{d^2x}{dt^2} + kx = 0$, with t in seconds.

a) Find the spring constant k and the natural angular frequency ω.

b) The mass is pushed 0.1 m above equilibrium and released with an upward velocity of 0.7 m/s. Find $x(t)$ and the amplitude of the motion.

c) Find the frequency of the motion, in oscillations per second.

Free Damped Motion and the Three Damping Cases

3. A mass of 1 kg hangs from a spring with spring constant 9 N/m and is attached to a dashpot that exerts a force opposite to the velocity with damping constant β, in N $\cdot$ s/m. With no external force the displacement $x(t)$ satisfies $m\dfrac{d^2x}{dt^2} + \beta\dfrac{dx}{dt} + kx = 0$. For each value of β, say whether the motion is overdamped, critically damped or underdamped, and write the general solution.

a) $\beta = 10$

b) $\beta = 6$

c) $\beta = 2$

4. A mass of 0.5 kg hangs from a spring with spring constant 10 N/m, with a damping force of 2 N · s/m times the velocity, opposite to it. The displacement $x(t)$, in metres below equilibrium, satisfies $m\dfrac{d^2x}{dt^2} + \beta\dfrac{dx}{dt} + kx = 0$, with t in seconds. At $t = 0$ the mass is 0.3 m below equilibrium and moving downward at 1 m/s.

a) Solve for $x(t)$.

b) Write the solution as $x = Ae^{-\lambda t}\cos(\mu t - \delta)$ with $A > 0$, and state the quasi-period.

c) The factor $Ae^{-\lambda t}$ bounds the oscillation. When has it fallen to 0.05 m?

Driven Motion with Transient and Steady-State Terms

5. A mass of 1 kg on a spring with spring constant 13 N/m and damping constant 4 N·s/m is driven by an external force $F(t) = 4\cos t$ newtons. The displacement $x(t)$, in metres, satisfies $m\dfrac{d^2x}{dt^2} + \beta\dfrac{dx}{dt} + kx = F(t)$, with t in seconds.

a) Find a particular solution of the form $x_p = A\cos t + B\sin t$ by undetermined coefficients.

b) Write the general solution, and say which part is the transient and which the steady state.

c) Find the amplitude of the steady-state motion.

6. A mass of 1 kg on a spring with spring constant 2 N/m and damping constant 3 N · s/m starts at rest at its equilibrium position. From $t = 0$ it is driven by the force $F(t) = 2\sin t$ newtons. The displacement $x(t)$, in metres, satisfies $m\dfrac{d^2x}{dt^2} + \beta\dfrac{dx}{dt} + kx = F(t)$, with t in seconds.

a) Solve the initial value problem.

b) Identify the transient and the steady-state terms, and give the amplitude of the steady-state motion.

Resonance

7. A mass of 2 kg hangs from a spring with spring constant 18 N/m, with no damping. It starts at rest at equilibrium and is driven by a force $F(t)$ newtons, so its displacement $x(t)$, in metres, satisfies $m\dfrac{d^2x}{dt^2} + kx = F(t)$, with t in seconds.

a) Find the natural angular frequency of the spring.

b) Solve for $x(t)$ when $F(t) = 12\cos 3t$, and describe the motion as t grows.

c) Solve for $x(t)$ when $F(t) = 12\cos 2t$ instead, and say whether the motion stays bounded.

Series Circuits with Resistance, Inductance and Capacitance

8. A series circuit has an inductor of 0.5 H, a resistor of 4 Ω and a capacitor of 0.025 F, and no source. By Kirchhoff's voltage law the charge $q(t)$ on the capacitor, in coulombs, satisfies $L\dfrac{d^2q}{dt^2}+R\dfrac{dq}{dt}+\dfrac{1}{C}\,q=E(t)$, and the current is $i=\dfrac{dq}{dt}$ amperes, with t in seconds. At $t=0$ the charge is 0.02 C and no current flows.

a) Is the circuit overdamped, critically damped or underdamped?

b) Find $q(t)$ and $i(t)$.

9. A series circuit has an inductor of 1 H, a resistor of 5 Ω, a capacitor of 0.25 F and an alternating source $E(t)=34\cos t$ volts. The charge $q(t)$, in coulombs, satisfies $L\dfrac{d^2q}{dt^2}+R\dfrac{dq}{dt}+\dfrac{1}{C}\,q=E(t)$, and the current is $i=\dfrac{dq}{dt}$ amperes, with t in seconds.

a) Find the steady-state charge and the steady-state current.

b) Write the general solution for $q(t)$ and explain why the initial charge and current have no effect in the long run.

Synthesis — drawing on several topics in this unit

10. A series circuit has an inductor of 2 H, a resistor of 8 Ω and a capacitor of 0.1 F. A 20 V battery is switched in at $t = 0$, when the capacitor is uncharged and no current flows. The charge $q(t)$, in coulombs, satisfies $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{1}{C}\,q = E(t)$, and the current is $i = \dfrac{dq}{dt}$ amperes, with t in seconds.

a) Classify the circuit as overdamped, critically damped or underdamped.

b) Find $q(t)$ and $i(t)$.

c) Identify the steady-state charge and the transient part.