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# The Laplace Transform

Name: ______________________________ Date: ______________

## Concepts

- Laplace Transforms from the Definition and from a Table
- Inverse Transforms and Partial Fractions
- Transforms of Derivatives and Initial Value Problems
- Translation on the s-Axis (theory)
- The Unit Step Function and Translation on the t-Axis
- Initial Value Problems with Piecewise and Periodic Forcing
- The Convolution Theorem (theory)
- The Dirac Delta Function and Impulsive Forcing

### Laplace Transforms from the Definition and from a Table

**1.** The Laplace transform of a function $f$ defined for $t \geq 0$ is $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$, for those $s$ at which the improper integral converges. Use this definition to find $F(s)$ for

$$f(t) = \begin{cases} t, & 0 \leq t < 1, \\ 1, & t \geq 1, \end{cases}$$

and state the real values of $s$ for which it exists.

**2.** Use linearity and the pairs

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad \mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \quad \mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$$

to find the Laplace transform of each function.

**a)** $f(t) = 3t^2 - 4e^{-t} + 2\sin 5t$

**b)** $g(t) = \left(1 + e^{2t}\right)^2$

**c)** $h(t) = \cos^2 2t$

### Inverse Transforms and Partial Fractions

**3.** Using the pairs

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad \mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \quad \mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2},$$

find each inverse transform.

**a)** $\mathcal{L}^{-1}\left\{\frac{4}{s^3} - \frac{6}{s-5}\right\}$

**b)** $\mathcal{L}^{-1}\left\{\frac{2s+7}{s^2+9}\right\}$

**4.** Using the pairs

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \quad \mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2},$$

find $\mathcal{L}^{-1}\left\{\dfrac{5s^2+3s+16}{s\,(s^2+4)}\right\}$.

**Transforms of Derivatives and Initial Value Problems**

**5.** Here $y = y(t)$. Using $\mathcal{L}\{y'\} = sY(s) - y(0)$, $\mathcal{L}\{y''\} = s^2Y(s) - s\,y(0) - y'(0)$ and $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$, solve the initial value problem

$$y'' - y' - 6y = 0, \qquad y(0) = 1, \quad y'(0) = 8$$

by the Laplace transform.

**6.** Here $y = y(t)$. Using $\mathcal{L}\{y''\} = s^2Y(s) - s\,y(0) - y'(0)$ and

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \quad \mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \quad \mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2},$$

solve $y'' + 4y = 8t,\ y(0) = 1,\ y'(0) = 0$ by the Laplace transform.

### Translation on the s-Axis

**7.** The first translation theorem says: if $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$. Using it together with

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \quad \mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2},$$

**a)** find $\mathcal{L}\{t^2e^{-3t}\}$;

**b)** find $\mathcal{L}^{-1}\left\{\dfrac{s+5}{s^2+4s+13}\right\}$.

### The Unit Step Function and Translation on the t-Axis

**8.** The unit step function is $u(t-a) = 0$ for $t < a$ and 1 for $t \geq a$, where $a \geq 0$, and $\mathcal{L}\{u(t-a)\} = \dfrac{e^{-as}}{s}$. Write

$$f(t) = \begin{cases} 2, & 0 \leq t < 3, \\ -1, & 3 \leq t < 5, \\ 0, & t \geq 5, \end{cases}$$

as a combination of unit step functions, and find $\mathcal{L}\{f(t)\}$.

**9.** The second translation theorem says: if $\mathcal{L}\{f(t)\} = F(s)$ and $a \geq 0$, then $\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$. Using it with $\mathcal{L}\{1\} = \dfrac{1}{s}$, $\mathcal{L}\{t^2\} = \dfrac{2}{s^3}$ and $\mathcal{L}\{e^{ct}\} = \dfrac{1}{s-c}$,

**a)** find $\mathcal{L}\{(t-2)^2\,u(t-2)\}$;

**b)** find $\mathcal{L}^{-1}\left\{\dfrac{3e^{-4s}}{s(s+3)}\right\}$, and write the answer piecewise.

### Initial Value Problems with Piecewise and Periodic Forcing

**10.** Here $y = y(t)$ and $u(t-a)$ is the unit step (0 for $t < a$, 1 for $t \geq a$). Use $\mathcal{L}\{y''\} = s^2 Y - s\,y(0) - y'(0)$, $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ where $F = \mathcal{L}\{f\}$, and

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \quad \mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$$

to solve

$$y'' + 4y = 8\,u(t-\pi), \qquad y(0) = 0, \quad y'(0) = 2.$$

Give the solution piecewise.

**11.** Here $y = y(t)$. If $f$ is periodic with period $T$, then $\mathcal{L}\{f(t)\} = \dfrac{1}{1-e^{-Ts}} \displaystyle\int_0^T e^{-st} f(t)\,dt$. Let $f$ be the square wave of period 2 with $f(t) = 1$ for $0 \leq t < 1$ and $f(t) = 0$ for $1 \leq t < 2$.

**a)** Find $\mathcal{L}\{f(t)\}$ and simplify it.

**b)** Using $\mathcal{L}\{y'\} = sY - y(0)$, find the transform $Y(s)$ of the solution of $y' + 2y = f(t)$, $y(0) = 0$. Do not invert it.

### The Convolution Theorem

**12.** The convolution of $f$ and $g$ is $(f * g)(t) = \displaystyle\int_0^t f(\tau)\, g(t-\tau)\, d\tau$, and the convolution theorem says $\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$.

**a)** Compute $t * e^t$ from the definition.

**b)** Check your answer by finding its inverse transform another way: invert $\mathcal{L}\{t\} \cdot \mathcal{L}\{e^t\} = \dfrac{1}{s^2(s-1)}$ by partial fractions, using $\mathcal{L}\{1\} = \dfrac{1}{s}$, $\mathcal{L}\{t\} = \dfrac{1}{s^2}$, $\mathcal{L}\{e^t\} = \dfrac{1}{s-1}$.

### The Dirac Delta Function and Impulsive Forcing

**13.** Here $y = y(t)$, $\delta(t-a)$ is the Dirac delta at $a > 0$, with $\mathcal{L}\{\delta(t-a)\} = e^{-as}$, and $u(t-a)$ is the unit step (0 for $t < a$, 1 for $t \geq a$). Using $\mathcal{L}\{y'\} = sY - y(0)$, $\mathcal{L}\{e^{ct}\} = \dfrac{1}{s-c}$ and $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$, solve

$$y' + 3y = 2\,\delta(t-1), \qquad y(0) = 4,$$

and state how the impulse changes $y$ at $t = 1$.

**14.** Here $y = y(t)$, $\delta(t-a)$ is the Dirac delta at $a > 0$, with $\mathcal{L}\{\delta(t-a)\} = e^{-as}$, and $u(t-a)$ is the unit step. Using $\mathcal{L}\{y''\} = s^2Y - s\,y(0) - y'(0)$, $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$, $\mathcal{L}\{e^{ct}\cos t\} = \dfrac{s-c}{(s-c)^2+1}$ and $\mathcal{L}\{e^{ct}\sin t\} = \dfrac{1}{(s-c)^2+1}$, solve

$$y'' + 2y' + 2y = \delta(t-2), \qquad y(0) = 1, \quad y'(0) = -1.$$

**Synthesis — drawing on several topics in this unit**

**15.** Here $y = y(t)$ and $u(t-a)$ is the unit step (0 for $t < a$, 1 for $t \geq a$). Using $\mathcal{L}\{y'\} = sY - y(0)$, $\mathcal{L}\{y''\} = s^2Y - s\,y(0) - y'(0)$, $\mathcal{L}\{1\} = \dfrac{1}{s}$, $\mathcal{L}\{e^{ct}\} = \dfrac{1}{s-c}$ and $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$, solve

$$y'' + 3y' + 2y = 2\,u(t-1), \qquad y(0) = 0, \quad y'(0) = 1,$$

and find $\lim\limits_{t\to\infty} y(t)$.