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Power Series Solutions

Name: ______________________ Date: ____________

Concepts

- Shifting the Index of Summation (theory)
- Series Solutions About an Ordinary Point
- The Recurrence Relation and the First Terms of a Solution
- Interval of Convergence and Estimating a Solution at a Point (theory)

Shifting the Index of Summation

1. Write

$$S = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 4\sum_{n=0}^{\infty} c_n x^{n+1}$$

as a single series of the form $\sum_k (\cdots)\, x^k$, in which every term carries the power x^k. Any term that does not fit the common range of k should be written separately in front of the sum.

Series Solutions About an Ordinary Point

2. A point x_0 is an *ordinary point* of $P(x)y'' + Q(x)y' + R(x)y = 0$ when, after dividing by $P(x)$ to get $y'' + p(x)y' + q(x)y = 0$, both p and q are analytic at x_0, that is, each is given by a convergent power series about x_0. Otherwise x_0 is a *singular point*. For each equation, with $y = y(x)$, find every real singular point and decide whether $x_0 = 0$ is an ordinary point.

a) $(x^2 + 3)y'' - xy' + y = 0$

b) $(x^2 - 2x)y'' + 3y' - y = 0$

c) $y'' + e^x y' + (\cos x)\, y = 0$

3. Consider $(x^2+2)y'' + 3xy' - 3y = 0$, where $y = y(x)$. The point $x = 0$ is an ordinary point.

a) Substitute $y = \sum_{n=0}^{\infty} c_n x^n$, write the result as a single series in x^n, and show that the coefficients satisfy

$$c_{n+2} = -\frac{(n-1)(n+3)}{2(n+2)(n+1)}\, c_n, \qquad n \geq 0.$$

b) Show that one solution is a polynomial, and find it.

c) Find the first four nonzero terms of a second solution y_2 with $y_2(0) = 1$ and $y_2'(0) = 0$, and write the general solution.

The Recurrence Relation and the First Terms of a Solution

4. Solve the initial-value problem

$$y'' - (1+x)y = 0, \qquad y(0) = 2, \quad y'(0) = -1,$$

where $y = y(x)$, by a power series about $x = 0$. Find the recurrence relation for the coefficients, and write the solution through the term in x^5.

5. The equation $xy'' + y = 0$, where $y = y(x)$, has a singular point at $x = 0$, but $x = 1$ is an ordinary point. Find the first four nonzero terms of the solution of

$$xy'' + y = 0, \qquad y(1) = 0, \quad y'(1) = 2,$$

as a series in powers of $(x-1)$. (Write $t = x - 1$ and $x = t + 1$.)

Interval of Convergence and Estimating a Solution at a Point

6. Use the following theorem. *If P, Q, R are polynomials with no common factor and x_0 is an ordinary point of $P(x)y'' + Q(x)y' + R(x)y = 0$, then every power series solution about x_0 converges at least for $|x - x_0| < \rho$. Here ρ is the distance, in the complex plane, from x_0 to the nearest zero of P.* Consider

$$(x^2 - 4x + 13)\, y'' + xy' - 4y = 0, \qquad y = y(x).$$

a) Find the singular points of the equation.

b) Give the interval on which the theorem guarantees convergence of a power series solution about $x_0 = 0$, about $x_0 = 2$, and about $x_0 = 6$.

Synthesis — drawing on several topics in this unit

7. Consider the initial-value problem

$$y'' + xy' + 2y = 0, \qquad y(0) = 0, \quad y'(0) = 1,$$

where $y = y(x)$.

a) Substitute $y = \sum_{n=0}^{\infty} c_n x^n$, shift indices so that every sum carries x^n, and find the recurrence relation.

b) Find the coefficients of the solution and a formula for the general nonzero coefficient.

c) On what interval does the series converge? Justify your answer.

d) Estimate $y(1)$ with an error smaller than 10^{-3}. Use the fact that if the terms of a series alternate in sign and decrease in absolute value to 0, then the error in stopping is at most the absolute value of the first term left out.