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Systems of Linear Differential Equations

Name: ______________________________ Date: ______________

Concepts

- Writing a Higher-Order Equation as a First-Order System
- Linear Systems in Matrix Form and the General Solution (theory)
- Homogeneous Systems with Distinct Real Eigenvalues
- Homogeneous Systems with Complex Eigenvalues
- Homogeneous Systems with a Repeated Eigenvalue (theory)
- Nonhomogeneous Linear Systems

Writing a Higher-Order Equation as a First-Order System

1. Consider the third-order equation

$$y''' - 2y'' + t\,y' - 5y = \cos t, \qquad y(0) = 1,\ y'(0) = 0,\ y''(0) = -2,$$

where $y = y(t)$. Introduce $x_1 = y$, $x_2 = y'$, $x_3 = y''$ and write the problem as a first-order system $\vec{x}\,' = A(t)\,\vec{x} + \vec{f}(t)$ with an initial condition $\vec{x}(0)$. Give $A(t)$, $\vec{f}(t)$ and $\vec{x}(0)$ explicitly.

2. The functions $u = u(t)$ and $v = v(t)$ satisfy the coupled pair

$$u'' + 3u - v = 0, \qquad v'' - 2u + 2v = t.$$

a) Write the pair as one first-order system $\vec{x}\,' = A\vec{x} + \vec{f}(t)$, stating what each component of $\vec{x}$ stands for.

b) How many scalar initial conditions does an initial-value problem for this pair need, and which ones?

Linear Systems in Matrix Form and the General Solution

3. Let $A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$, $\vec{x}_1(t) = e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\vec{x}_2(t) = e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

a) Verify that $\vec{x}_1$ and $\vec{x}_2$ are solutions of $\vec{x}\,' = A\vec{x}$.

b) Compute the Wronskian $W(\vec{x}_1, \vec{x}_2)(t)$ and explain why the two solutions form a fundamental set on $(-\infty, \infty)$.

c) Write the general solution, then find the solution with $\vec{x}(0) = (3, -4)$.

Homogeneous Systems with Distinct Real Eigenvalues

4. Use the eigenvalues and eigenvectors of the coefficient matrix to find the general solution of

$$\vec{x}\,' = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \vec{x}.$$

5. Solve the initial-value problem

$$\vec{x}\,' = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

Homogeneous Systems with Complex Eigenvalues

6. Find the general real-valued solution of

$$\vec{x}\,' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \vec{x}.$$

7. Solve the initial-value problem

$$\vec{x}\,' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Homogeneous Systems with a Repeated Eigenvalue

8. The matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ has the single eigenvalue $\lambda = 2$.

a) Show that every eigenvector of A is a multiple of $(1, 1)$.

b) Find a vector $\vec{w}$ with $(A - 2I)\vec{w} = (1, 1)$, and write the general solution of $\vec{x}\,' = A\vec{x}$ using the solutions $e^{2t}\vec{v}$ and $e^{2t}(t\vec{v} + \vec{w})$, where $\vec{v} = (1, 1)$.

c) Solve the system with $\vec{x}(0) = (3, 1)$.

Nonhomogeneous Linear Systems

9. Consider $\vec{x}\,' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} -2t \\ -t - 2 \end{bmatrix}$.

a) Use undetermined coefficients with a trial solution $\vec{x}_p = t\,\vec{a} + \vec{b}$, where $\vec{a}$ and $\vec{b}$ are constant vectors, to find a particular solution.

b) Write the general solution.

10. Use variation of parameters to find a particular solution of

$$\vec{x}\,' = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ e^t \end{bmatrix},$$

and write the general solution.

Synthesis — drawing on several topics in this unit

11. Consider $y'' - 3y' - 4y = 0$, where $y = y(t)$, with $y(0) = 2$ and $y'(0) = 3$.

a) Write the problem as a first-order system $\vec{x}\,' = A\vec{x}$ with $x_1 = y$, $x_2 = y'$, and give $\vec{x}(0)$.

b) Solve the system using the eigenvalues and eigenvectors of A, and read off $y(t)$.