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Differential Equations and Direction Fields

Name: ______________________________ Date: ______________

Concepts

- Order, Linearity and Classification of an Equation
- General Solutions, Particular Solutions and Families of Curves (theory)
- Direction Fields and Solution Curves
- Autonomous Equations and the Stability of an Equilibrium (theory)
- Existence and Uniqueness for an Initial Value Problem (theory)
- Euler's Method

Order, Linearity and Classification of an Equation

1. For each equation, state (i) its order, (ii) the independent and the dependent variable, and (iii) whether it is linear or nonlinear, naming a term responsible when it is nonlinear. For each linear equation, say also whether it is homogeneous.

a) $x^3y''' - 2xy' + 5y = e^x$, where $y = y(x)$

b) $\dfrac{d^2u}{dt^2} + 4\sin u = 0$

c) $(1 - y)\, y' + 2y = x$, where $y = y(x)$

d) $\dfrac{dx}{dt} = t^2x + \cos t$

e) $y'' = \sqrt{1 + (y')^2}$, where $y = y(x)$

2. Let a, b be real constants and c a non-negative integer, and consider

$$y'' + (a-2)\,y\,y' + b\,y^c = \cos x, \qquad y = y(x),$$

where y^0 means the constant 1, whatever the value of y. Find every choice of a, b and c for which this equation is linear. For those choices, is it ever homogeneous?

General Solutions, Particular Solutions and Families of Curves

3. Consider $xy' + y = 3x^2$, where $y = y(x)$.

a) Show that $y = x^2 + \dfrac{C}{x}$ is a solution on $(0, \infty)$ for every constant C.

b) Find the particular solution with $y(1) = 4$ and state the largest interval on which it is a solution.

c) Exactly one member of the family is a solution on the whole real line. Which one, and why only that one?

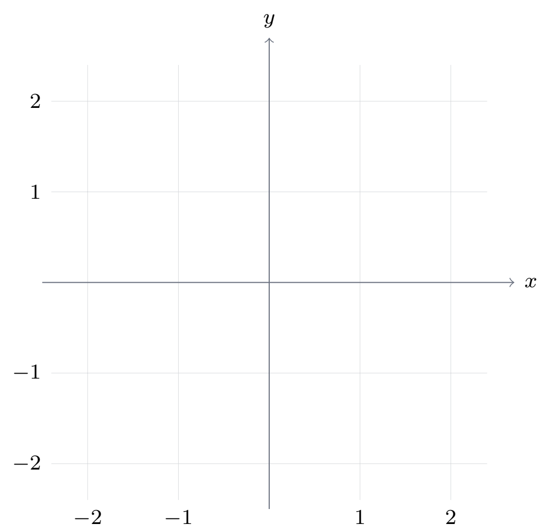
Direction Fields and Solution Curves

4. Consider $y' = 2x - y$, where $y = y(x)$.

a) Compute the slope of the direction field at the nine points (x, y) with $x, y \in \{-1, 0, 1\}$, and draw the nine short segments on the grid.

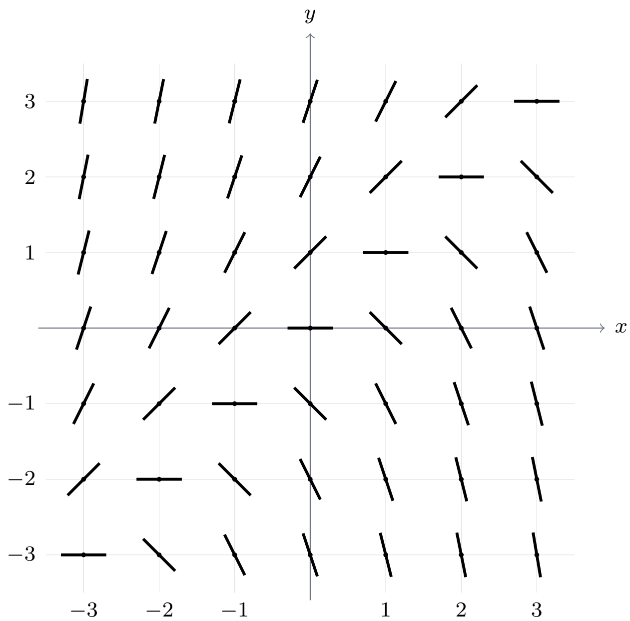
b) On which line are all the segments horizontal?

c) Show that $y = 2x - 2$ is a solution, and explain why every segment centred on that line points along it.



5. The direction field below was drawn for exactly one of

$$\text{(A) } y' = x - y, \qquad \text{(B) } y' = xy, \qquad \text{(C) } y' = y - x, \qquad \text{(D) } y' = x + y.$$



a) Which equation is it? For each of the other three, name one grid point where its slope differs from the drawn one.

b) One solution curve is a straight line. Find it and check it.

c) Sketch the solutions through $(0, 0)$ and $(0, 2)$ and describe what each does as $x \to \infty$.

Autonomous Equations and the Stability of an Equilibrium

6. Consider the autonomous equation

$$\frac{dy}{dt} = y^2(y-3)(y+2).$$

a) Find the equilibrium solutions and draw the phase line.

b) Classify each equilibrium as stable, unstable or semistable.

c) Give $\lim_{t\to\infty} y(t)$ for the solutions with $y(0) = 1$ and with $y(0) = -1$.

Existence and Uniqueness for an Initial Value Problem

7. The existence and uniqueness theorem says: if f and $\dfrac{\partial f}{\partial y}$ are both continuous on an open rectangle $a < x < b$, $c < y < d$ containing (x_0, y_0), then the initial value problem $y' = f(x, y),\ y(x_0) = y_0$ has exactly one solution on some open interval containing x_0.

Apply it to

$$y' = \frac{\sqrt[3]{y-2}}{x+1},$$

where $\sqrt[3]{\ }$ is the real cube root.

a) Find every initial point (x_0, y_0) at which the theorem guarantees a unique solution.

b) Say what the theorem tells you at each of $(0, 5)$, $(0, 2)$, $(-1, 3)$ and $(3, -6)$.

Euler's Method

8. Euler's method with step h computes $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h\,f(x_n, y_n)$ for the problem $y' = f(x, y),\ y(x_0) = y_0$.

a) Use two steps with $h = 0.5$ to approximate $y(1)$ for $y' = x^2 - y,\ y(0) = 2$.

b) Verify that $y = x^2 - 2x + 2$ is the exact solution of this initial value problem, and find the error in your approximation.

9. For the initial value problem $y' = y^2 - x,\ y(1) = 1$, use Euler's method ($x_{n+1} = x_n + h,\ y_{n+1} = y_n + h\,f(x_n, y_n)$) with $h = -0.5$.

a) Approximate $y(0)$ in two steps.

b) By finding the sign of y'' at $(1, 1)$, decide whether the first step overestimates or underestimates $y(0.5)$.

Synthesis — drawing on several topics in this unit

10. The existence and uniqueness theorem says: if f and $\dfrac{\partial f}{\partial y}$ are both continuous on an open rectangle containing (x_0, y_0), then $y' = f(x, y)$, $y(x_0) = y_0$ has exactly one solution on some open interval containing x_0. Consider

$$xy' = 2y, \qquad y = y(x).$$

a) State its order and whether it is linear and homogeneous.

b) Show that $y = Cx^2$ is a solution on all of $\mathbb{R}$ for every constant C.

c) Show that no solution has $y(0) = 1$, and that infinitely many have $y(0) = 0$.

d) Explain why (c) does not contradict the theorem, and say at which points (x_0, y_0) the theorem does guarantee a unique solution.