UNIVERSITY DIFFERENTIAL EQUATIONS · PRACTICE WORKSHEET · TUTORINMONTREAL.CA

First-Order Solution Methods

Name: ______________________________________ Date: _________________

Concepts

- Separable Equations with Implicit and Lost Solutions (theory)
- Linear First-Order Equations and the Integrating Factor
- Exact Equations
- Integrating Factors for Equations That Are Not Exact (theory)
- Homogeneous Equations Solved by Substitution
- Bernoulli Equations
- Choosing a Method for a First-Order Equation (theory)

Separable Equations with Implicit and Lost Solutions

1. Consider the initial value problem

$$\frac{dy}{dx} = \frac{2x+1}{3y^2+e^y}, \qquad y(0) = 0.$$

a) Solve it, leaving the solution in implicit form.

b) Verify your answer by implicit differentiation, and explain why the solution is left in implicit form rather than solved for y.

Linear First-Order Equations and the Integrating Factor

2. Let $y = y(t)$. Use an integrating factor to solve

$$y' + 4y = 8t\,e^{-4t},$$

first in general and then with the initial condition $y(0) = 3$.

3. Solve the initial value problem

$$x\,y' + 3y = 10x^2, \qquad y(1) = 5,$$

where $y = y(x)$, and state the largest interval on which the solution is valid.

Exact Equations

4. Show that the equation

$$\left(2xy^3 + \sin y\right) dx + \left(3x^2y^2 + x\cos y\right) dy = 0$$

is exact, and find its general solution in implicit form.

5. Find the value of the constant k for which

$$\left(6xy + y^3\right) dx + \left(kx^2 + 3xy^2 + 1\right) dy = 0$$

is exact. For that value, solve the equation with the initial condition $y(1) = 1$.

Integrating Factors for Equations That Are Not Exact

6. Consider

$$\left(3xy + 2y^2\right) dx + \left(x^2 + 2xy\right) dy = 0.$$

Show that the equation is not exact, find an integrating factor that depends on x alone, and solve the equation.

Homogeneous Equations Solved by Substitution

7. Use the substitution $y = ux$ to find the general solution of

$$xy \frac{dy}{dx} = x^2 + 2y^2, \qquad x > 0.$$

8. Solve the initial value problem

$$x^2\, y' = x^2 + xy + y^2, \qquad y(1) = 0,$$

where $y = y(x)$, by the substitution $y = ux$. Give y explicitly and state the largest interval on which the solution is valid.

Bernoulli Equations

9. Let $y = y(x)$ with $x > 0$. Solve the Bernoulli equation

$$y' + \frac{1}{x}\, y = x^2 y^3$$

by the substitution $v = y^{-2}$, and state the solution that the substitution loses.

10. Solve the initial value problem

$$y' - y = e^x y^2, \qquad y(0) = 1,$$

where $y = y(x)$. Give y explicitly and state the largest interval on which the solution is valid.

Choosing a Method for a First-Order Equation

11. For each equation, where $y = y(x)$, list every one of the following classes it belongs to: *separable*, *linear in* y, *homogeneous* (the right side of $y' = f(x,y)$ is a function of y/x alone), *Bernoulli* ($y' + p(x)y = q(x)y^n$ with $n \neq 0, 1$). For (b), which is given in differential form, also decide whether it is *exact* as written. Justify each answer in a line; do not solve.

a) $\dfrac{dy}{dx} = \dfrac{x^3 + y^3}{xy^2}$

b) $(y \cos x + 2x)\, dx + \big(\sin x - 3y^2\big)\, dy = 0$

c) $\dfrac{dy}{dx} = x\, e^{-x} y^2$

d) $\dfrac{dy}{dx} = \dfrac{2y}{x} + x^2$

Synthesis — drawing on several topics in this unit

12. Consider the initial value problem

$$2xy\,y' = 3y^2 - x^2, \qquad y(1) = 2,$$

where $y = y(x)$ and $x > 0$.

a) Solve it by the substitution $y = ux$.

b) Solve it again as a Bernoulli equation, with the substitution $v = y^2$, and confirm that the two answers agree.

c) Give y explicitly and state its interval of validity.