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Modelling with First-Order Equations

Name: ______________________ Date: ______________

Concepts

- Translating a Rate Statement into a Differential Equation
- Newton's Law of Cooling and Heating (theory)
- Mixing Problems with Unequal Flow Rates
- First-Order Electric Circuits
- The Logistic Equation (theory)

Translating a Rate Statement into a Differential Equation

1. For each statement, name the unknown function and its independent variable, write the differential equation it describes (use k for the constant of proportionality and say what sign it has), and say whether the equation is linear or nonlinear.

a) On a frozen pond, the thickness h of the ice, in centimetres, increases with time t, in hours, at a rate inversely proportional to the thickness already present.

b) In a town of 8000 residents, the number who have heard a rumour increases with time t, in days, at a rate jointly proportional to the number who have heard it and the number who have not.

c) Sunlight entering murky water loses intensity with depth: the intensity I decreases with the depth x below the surface, in metres, at a rate proportional to the intensity at that depth.

2. Dead leaves fall on a patch of forest floor at a constant rate of 4 kg/m^2 per year, and the litter on the ground decomposes at a rate proportional to the amount present, with constant of proportionality 0.5 per year. The patch is cleared completely at $t = 0$. Let $L(t)$ be the litter on the ground, in kg/m^2, t years later.

a) Write the initial-value problem for L.

b) Give the units of $\dfrac{dL}{dt}$ and of each term on the right-hand side, and confirm that they agree.

c) Without solving the equation, find the amount of litter the patch approaches in the long run, and explain why that value is the one to expect.

Newton's Law of Cooling and Heating

3. A steel bolt at 840 °C is plunged into a large oil bath held at 40 °C. Newton's law of cooling says that the rate of change of the bolt's temperature is proportional to the difference between the bath temperature and the bolt's temperature. Two minutes later the bolt is at 240 °C. Let $T(t)$ be its temperature, in °C, t minutes after it enters the bath.

a) Write the initial-value problem and solve it, finding the constant exactly.

b) When does the bolt reach 90 °C?

Mixing Problems with Unequal Flow Rates

4. A hydroponic reservoir can hold 100 L. It starts with 50 L of water in which 10 g of nutrient is dissolved. A solution containing 1 g/L of nutrient flows in at 3 L/min, the reservoir is kept well stirred, and the mixture is pumped out at 2 L/min. Let $x(t)$ be the mass of nutrient in the reservoir, in grams, after t minutes.

a) Find the volume $V(t)$ and write the initial-value problem for x.

b) Solve it.

c) How much nutrient is in the reservoir at the moment it starts to overflow, and what is the concentration then?

5. A 120 L tank holds a solution containing 6 kg of disinfectant. A solution with 0.1 kg/L of disinfectant flows in at 2 L/min, and the well-stirred mixture drains at 5 L/min. Let $x(t)$ be the mass of disinfectant in the tank, in kilograms, after t minutes. Do not solve the equation.

a) Find the volume $V(t)$ and the time at which the tank is empty.

b) Write the initial-value problem for x, and state the interval of t on which the model applies.

c) Is the equation separable? Is it linear? Justify both answers.

First-Order Electric Circuits

6. A series circuit has a resistor of 10 Ω, an inductor of 2 H and a 40 V battery switched in at $t = 0$, when no current flows. Kirchhoff's voltage law gives $L\dfrac{di}{dt} + Ri = E$, where $i(t)$ is the current in amperes and t is in seconds.

a) Solve for $i(t)$ by an integrating factor.

b) State the steady-state current and the time at which the current reaches 3 A.

7. A series circuit has a resistor of 20 Ω, a capacitor of 0.01 F and a 50 V source. When the switch closes at $t = 0$ the capacitor already carries a charge of 0.1 C. The charge $q(t)$, in coulombs, satisfies $R\dfrac{dq}{dt} + \dfrac{q}{C} = E$, and the current is $i = \dfrac{dq}{dt}$; t is in seconds.

a) Find $q(t)$ and $i(t)$.

b) When does the charge reach 0.3 C?

The Logistic Equation

8. A bird population on an island, $P(t)$ birds after t years, follows the logistic equation

$$\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{1200}\right), \qquad P(0) = 200.$$

a) Solve for $P(t)$.

b) When is the population growing fastest, and at what rate is it growing then?

Synthesis — drawing on several topics in this unit

9. An insulated 100 L hot-water tank is full of water at 70 °C. From $t = 0$, cold water at 10 °C flows in at 5 L/min, the tank is kept well mixed, and water leaves at 5 L/min. Treat the heat content of the tank as proportional to volume times temperature, and assume the rate of change of the heat content is the heat carried in minus the heat carried out; t is in minutes.

a) Show that the tank temperature $T(t)$ satisfies $\dfrac{dT}{dt} = \dfrac{1}{20}(10 - T)$, and point out which law of heat transfer the equation has the same form as.

b) Solve it, and find when the water leaving the tank has cooled to 40 °C.