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Homogeneous Linear Equations with Constant Coefficients

Name: ______________________ Date: ____________

Concepts

- Complex Numbers in Rectangular, Polar and Exponential Form
- Powers and Roots of a Complex Number (theory)
- Superposition, Linear Independence and the Wronskian (theory)
- Distinct Real Roots of the Characteristic Equation
- Repeated Roots of the Characteristic Equation
- Complex Conjugate Roots of the Characteristic Equation
- Homogeneous Equations of Third and Higher Order (theory)

Complex Numbers in Rectangular, Polar and Exponential Form

1. Let $z = 4 - 2i$ and $w = 1 + 3i$. Write each of the following in the form $a + bi$ with a and b real.

a) zw

b) $\dfrac{z}{w}$

c) $w^2 - \overline{z}$, where $\overline{z}$ is the complex conjugate of z

2. Throughout, the argument of a nonzero complex number is taken in $(-\pi, \pi]$.

a) Write $z = -2\sqrt{3} + 2i$ in polar form $r(\cos\theta + i\sin\theta)$ and in exponential form $re^{i\theta}$.

b) Write $6e^{-i\pi/4}$ in the form $a + bi$.

c) Use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to write $e^{(-1+2i)t}$, for real t, in the form $u(t) + i\,v(t)$ with u and v real-valued functions.

Powers and Roots of a Complex Number

3. Use polar or exponential form in each part.

a) Compute $(1 - i)^{10}$, giving the answer in the form $a + bi$.

b) Find all three cube roots of $-8i$, each in the form $a + bi$.

Superposition, Linear Independence and the Wronskian

4. Here $y = y(x)$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

a) Verify that $y_1 = e^x$ and $y_2 = \sinh x$ are both solutions of $y'' - y = 0$ on $(-\infty, \infty)$.

b) Compute the Wronskian $W(y_1, y_2)$, and say what its value tells you about y_1 and y_2.

c) Write the general solution of $y'' - y = 0$ in terms of y_1 and y_2. Then write the solution e^{-x} as a combination $c_1y_1 + c_2y_2$.

Distinct Real Roots of the Characteristic Equation

5. Find the general solution of each equation, where $y = y(x)$.

a) $y'' + y' - 6y = 0$

b) $2y'' - 7y' + 3y = 0$

c) $y'' - 5y' = 0$

6. Solve the initial value problem $y'' - y' - 2y = 0,\ y(0) = 2,\ y'(0) = -5$, where $y = y(t)$. Then find the value of $t > 0$ at which the solution is zero.

Repeated Roots of the Characteristic Equation

7. Find the general solution of each equation, where $y = y(x)$.

a) $y'' + 6y' + 9y = 0$

b) $4y'' - 4y' + y = 0$

8. Solve the initial value problem $y'' - 8y' + 16y = 0,\ y(0) = 3,\ y'(0) = 10$, where $y = y(t)$, and find where the solution crosses zero.

Complex Conjugate Roots of the Characteristic Equation

9. Find the general solution of each equation in real form, where $y = y(x)$.

a) $y'' + 2y' + 5y = 0$

b) $y'' + 16y = 0$

c) $9y'' - 6y' + 10y = 0$

10. Solve the initial value problem $y'' - 4y' + 13y = 0,\ y(0) = 1,\ y'(0) = 8$, where $y = y(t)$.

Homogeneous Equations of Third and Higher Order

11. Find the general solution of each equation in real form, where $y = y(x)$.

a) $y''' - 3y'' - 4y' + 12y = 0$

b) $y^{(4)} + 8y'' + 16y = 0$

c) $y''' + 8y = 0$

Synthesis — drawing on several topics in this unit

12. Consider $y^{(4)} + 16y = 0$, where $y = y(x)$.

a) Find the four fourth roots of -16, each in the form $a + bi$.

b) Write the general solution of the equation in real form.

c) Find the solution that satisfies $y(0) = 1$ and $y'(0) = 0$ and stays bounded as $x \to \infty$.