University Discrete Mathematics · Practice Worksheet · tutorinmontreal.ca

Methods of Proof

Name: ______________________ Date: ______________

Concepts

- Direct Proof
- Proof by Contraposition
- Proof by Contradiction
- Proof by Cases (theory)
- Proving an If and Only If Statement (theory)
- Existence Proofs and Counterexamples (theory)

Direct Proof

1. An integer n is *even* if $n = 2k$ for some integer k, and *odd* if $n = 2k + 1$ for some integer k. Give a direct proof of the following statement.

If m and n are odd integers, then $mn + m + n$ is odd.

2. For integers a and b, we write $a \mid b$ ("a divides b") when $b = ak$ for some integer k.

a) Check the statement below for $a = 3$, $b = 6$, $c = 12$.

b) Give a direct proof: *for all integers a, b, c, if $a \mid b$ and $b \mid c$, then $a^2 \mid bc$.*

Proof by Contraposition

3. Let n be an integer. Consider the statement *"if $5n + 3$ is even, then n is odd."*

a) Write its contrapositive and its converse.

b) Prove the statement by contraposition. (An integer is even if it equals $2k$, odd if it equals $2k + 1$, for some integer k; every integer is exactly one of the two.)

4. Let x and y be real numbers. Prove by contraposition: *if $x + y > 10$, then $x > 5$ or $y > 5$.* Begin by writing the contrapositive, negating the "or" correctly.

Proof by Contradiction

5. Each statement below is to be proved by contradiction.

a) *There is no largest odd integer.* Write the assumption the proof starts from.

b) *If a, b, c are integers and $a+b+c$ is odd, then at least one of a, b, c is odd.* Write the assumption the proof starts from.

c) *For every real number $x > 0$, $x + \frac{4}{x} \geq 4$.* Write the assumption the proof starts from, then complete the proof.

6. Prove by contradiction that *there are no integers a and b with $a^2 - 4b = 2$.* You may use the fact that if a^2 is even, then a is even.

Proof by Cases

7. An integer is even if it equals $2k$, and odd if it equals $2k+1$, for some integer k; every integer is one or the other. Prove by cases that *for every integer n, the number $n^2 + 3n + 5$ is odd.*

Proving an If and Only If Statement

8. Let x be a real number. Prove that

$$x^2 - 4x + 3 < 0 \quad \text{if and only if} \quad 1 < x < 3,$$

proving each direction separately and saying which direction is which.

Existence Proofs and Counterexamples

9. A *prime* is an integer $p > 1$ whose only positive divisors are 1 and p. Here $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$.

a) Disprove: *for every $n \in \mathbb{N}$, the number $2^n + 1$ is prime.*

b) Disprove: *for all real numbers a and b, $(a + b)^2 \geq a^2 + b^2$.*

c) Prove: *there exist distinct $m, n \in \mathbb{Z}^+$ with $m^n = n^m$.*

Synthesis — drawing on several topics in this unit

10. Every integer n can be written in exactly one of the forms $3k$, $3k + 1$ or $3k + 2$ with k an integer. We write $3 \mid n$ when $n = 3k$ for some integer k.

a) Prove by cases that for every integer n: if $n = 3k$ then $n^2 = 3q$ for some integer q, and if $n = 3k+1$ or $n = 3k + 2$ then $n^2 = 3q + 1$ for some integer q.

b) Deduce from (a), by contraposition, that for every integer n, if $3 \mid n^2$ then $3 \mid n$.