UNIVERSITY DISCRETE MATHEMATICS · PRACTICE WORKSHEET · TUTORINMONTREAL.CA

Sets, Functions and Relations

Name: ______________________________ Date: ______________

Concepts

- Set Notation, Subsets and Power Sets
- Set Operations and Venn Diagrams (theory)
- Proving Set Identities
- Injective, Surjective and Bijective Functions
- Composition and Inverse Functions (theory)
- Relations and Their Properties
- Equivalence Relations and Partitions
- Partial Orders and Hasse Diagrams (theory)

Set Notation, Subsets and Power Sets

1. Throughout, $\mathbb{N} = \{0, 1, 2, \ldots\}$. List the elements of each set and give its cardinality.

a) $A = \{n \in \mathbb{Z} \mid n^2 \leq 7\}$

b) $B = \{3k - 1 \mid k \in \mathbb{N},\ k < 4\}$

c) $C = \{x \in \mathbb{Q} \mid x^2 = 2\}$ (you may use the fact that $\sqrt{2}$ is irrational)

d) Write the set $D = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$ in set-builder notation.

2. Let $A = \{p, q, r\}$, where p, q and r are three different letters (none of them is itself a set), and recall that the power set $\mathcal{P}(A)$ is the set of all subsets of A.

a) List the elements of $\mathcal{P}(A)$ and give $|\mathcal{P}(A)|$.

b) Decide whether each statement is true or false, with a one-line reason.
(i) $\varnothing \in \mathcal{P}(A)$ (ii) $\{p\} \in A$ (iii) $\{p\} \subseteq A$ (iv) $\{\{p\}, \{q, r\}\} \subseteq \mathcal{P}(A)$
(v) $\{p, \{q\}\} \subseteq \mathcal{P}(A)$ (vi) $A \in \mathcal{P}(A)$

Set Operations and Venn Diagrams

3. Let the universe be $U = \{1, 2, \ldots, 12\}$ and let $A = \{n \in U \mid n \text{ is even}\}$, $B = \{n \in U \mid n \text{ is a multiple of } 3\}$ and $C = \{n \in U \mid n \leq 4\}$. List the elements of each set.

a) $A \cap B$

b) $B \cup C$

c) $A \setminus B$

d) $\overline{A \cup B}$ (the complement is taken in U)

e) $(A \cap C) \times (B \cap C)$

f) Write the set of elements of U that lie in *exactly one* of A and B using only A, B, $\cup$, $\cap$ and $\setminus$, then list it.

Proving Set Identities

4. Use a membership table (one row for each of the eight patterns of membership in A, B and C) to decide whether each of the following holds for all sets A, B and C. Where one fails, name a row that shows it.

a) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

b) $(A \cup B) \setminus C = A \cup (B \setminus C)$

5. Using the standard set laws, and naming the law at each step (De Morgan, distributive, complement, identity, and so on), simplify

$$\overline{A \cup B} \;\cup\; (\overline{A} \cap B),$$

where complements are taken in a universe U.

Injective, Surjective and Bijective Functions

6. A function $f\colon X \to Y$ is *injective* (one-to-one) when different inputs always give different outputs, *surjective* (onto) when every element of Y is an output, and *bijective* when it is both. Classify each function as bijective, injective only, surjective only, or neither.

a) $f\colon \{1,2,3,4\} \to \{w,x,y,z\}$ with $f(1)=x$, $f(2)=z$, $f(3)=w$, $f(4)=y$

b) $g\colon \{1,2,3,4\} \to \{w,x,y\}$ with $g(1)=w$, $g(2)=y$, $g(3)=x$, $g(4)=y$

c) $h\colon \{1,2,3\} \to \{w,x,y,z\}$ with $h(1)=z$, $h(2)=x$, $h(3)=w$

d) Explain why no function from a 4-element set to a 3-element set can be injective.

7. Decide whether each function is injective and whether it is surjective. Justify each "yes" in general and each "no" with specific elements. Here $\mathbb{N} = \{0,1,2,\dots\}$.

a) $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(n) = 3n+1$

b) $g\colon \mathbb{Z} \to \mathbb{N}$, $g(n) = |n|$

c) $h\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$, $h(m,n) = 2m+3n$

Composition and Inverse Functions

8. Let $X = \{1, 2, 3, 4\}$ and let $f, g \colon X \to X$ be given by

x	1	2	3	4
$f(x)$	3	1	4	2
$g(x)$	2	2	1	4

a) Give the tables of $g \circ f$ and $f \circ g$, and say whether they are equal.

b) Give the table of f^{-1}.

c) Explain why g has no inverse function.

Relations and Their Properties

9. A relation R on a set X is *reflexive* if $(x, x) \in R$ for every $x \in X$; *symmetric* if $(a, b) \in R$ implies $(b, a) \in R$; *antisymmetric* if $(a, b) \in R$ and $(b, a) \in R$ imply $a = b$; and *transitive* if $(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$. On $X = \{1, 2, 3, 4\}$ let

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (2, 3)\}.$$

Decide which of the four properties R has. For each one it lacks, name the pairs that show it.

10. For each relation on $\mathbb{Z}$, decide whether it is reflexive, symmetric, antisymmetric and transitive. Justify each "yes" in general and each "no" with specific integers.

a) $a\,R\,b$ if and only if $a \leq b + 1$

b) $a\,S\,b$ if and only if $ab > 0$

Equivalence Relations and Partitions

11. On the set $X = \{-3, -2, -1, 0, 1, 2, 3\}$ define $a\,R\,b$ if and only if $a^2 = b^2$.

a) Explain briefly why R is an equivalence relation.

b) List its equivalence classes and say how many there are.

12.

a) The collection $\{\{1,4\},\{2\},\{3,5,6\}\}$ is a partition of $\{1,2,3,4,5,6\}$. List the ordered pairs of the equivalence relation whose classes are these blocks, and give the number of pairs.

b) Explain why neither $\{\{1,2\},\{2,3\},\{4,5,6\}\}$ nor $\{\{1,2\},\{4\},\{5,6\}\}$ is a partition of $\{1,2,3,4,5,6\}$.

Partial Orders and Hasse Diagrams

13. On $S = \{2,3,4,6,8,9,12\}$ write $a \mid b$ when b is an integer multiple of a. The relation $\mid$ is a partial order on S.

a) List the pairs $a \mid b$ with $a \neq b$ for which no $c \in S$ other than a and b satisfies $a \mid c$ and $c \mid b$, and draw the Hasse diagram.

b) Give the minimal and the maximal elements, and say whether S has a least or a greatest element.

c) Give two elements that are incomparable.

Synthesis — drawing on several topics in this unit

14. Let $X = \{1, 2, \ldots, 8\}$ and define $f\colon X \to \{0, 1, 2\}$ by $f(n) = \lfloor n/3 \rfloor$, the greatest integer less than or equal to $n/3$. Define a relation R on X by $a\,R\,b$ if and only if $f(a) = f(b)$.

a) Is f injective? Is it surjective?

b) List the equivalence classes of R.

c) How many ordered pairs does R contain?

d) List the set $\{n \in X \mid f(n) \geq 1\}$.