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Divisibility and Primes

Name: _____ Date: _____

Concepts

- Divisibility and the Division Algorithm
- Primes and the Fundamental Theorem of Arithmetic (theory)
- The Greatest Common Divisor and the Euclidean Algorithm
- Bezout's Identity and the Extended Euclidean Algorithm

Divisibility and the Division Algorithm

1. The division algorithm says: for integers a and b with $b > 0$ there are unique integers q and r with $a = bq + r$ and $0 \le r < b$. For each dividend a below, with $b = 23$, find q and r, and say whether $23 \mid a$.

a) $a = 317$

b) $a = -317$

c) $a = -92$

2. An integer n leaves remainder 7 when divided by 9. Using $n = 9q + 7$ for some integer q, find the remainder when each of the following is divided by 9.

a) $4n + 5$

b) n^2

c) $-n$

Primes and the Fundamental Theorem of Arithmetic

3.

a) Decide whether 437 and 401 are prime. Test only the divisors you need, and say why testing those is enough.

b) Let $A = 2^3 \cdot 3^2 \cdot 5 \cdot 7^2$ and $B = 2 \cdot 3^4 \cdot 7^3 \cdot 11$. Write $\gcd(A, B)$ and $\operatorname{lcm}(A, B)$ as products of prime powers, and find the number of positive divisors of A.

Use: a positive integer d divides $n = p_1^{e_1} \cdots p_k^{e_k}$ exactly when $d = p_1^{f_1} \cdots p_k^{f_k}$ with $0 \le f_i \le e_i$ for every i, so n has $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$ positive divisors.

The Greatest Common Divisor and the Euclidean Algorithm

4. Use the Euclidean algorithm to find $\gcd(1573, 1144)$. Show each division.

5.

a) Use the Euclidean algorithm to find $\gcd(4389, 2002)$.

b) Use $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$ (for positive a, b) to find $\operatorname{lcm}(4389, 2002)$.

c) The gcd of two integers, not both zero, is the largest positive integer dividing both. Find $\gcd(-4389, 2002)$ and $\gcd(2002, 0)$.

Bezout's Identity and the Extended Euclidean Algorithm

6. Use the extended Euclidean algorithm to find $\gcd(391, 299)$ and integers x, y with $391x + 299y = \gcd(391, 299)$.

7. Bezout's identity says the integers of the form $ax + by$, with $x, y \in \mathbb{Z}$, are exactly the multiples of $\gcd(a, b)$.

a) Explain why $18x + 42y = 100$ has no integer solution.

b) Find one integer solution of $18x + 42y = 30$.

Synthesis — drawing on several topics in this unit

8.

a) Use the Euclidean algorithm to find $\gcd(1309, 1001)$, and write it in the form $1309x + 1001y$ with $x, y \in \mathbb{Z}$.

b) Given that $1001 = 7 \cdot 11 \cdot 13$, use your gcd to write 1309 as a product of primes, and find $\operatorname{lcm}(1309, 1001)$.