University Discrete Mathematics · Practice Worksheet · tutorinmontreal.ca

Induction and Recursion

Name: ______________________________ Date: ______________

Concepts

- Mathematical Induction for Summation Formulas
- Induction for Divisibility and Inequalities
- Strong Induction
- Recursively Defined Sequences
- Recursive Algorithms and Their Correctness (theory)

Mathematical Induction for Summation Formulas

1. The notation $\sum_{i=1}^{n}(4i-1)$ means the sum of $4i-1$ over $i=1,2,\dots,n$, that is $3+7+11+\cdots+(4n-1)$. Let $P(n)$ be the statement

$$\sum_{i=1}^{n}(4i-1)=n(2n+1), \qquad n\in\mathbb{Z}^{+}.$$

a) Write out the sum for $n=4$ term by term and check that $P(4)$ is true.

b) Write the statements $P(k)$ and $P(k+1)$.

c) Assuming $P(k)$, show that $P(k+1)$ follows. Say at which step you use $P(k)$.

d) Which value of n must the base case check, and why?

2. Prove by induction that for every $n \in \mathbb{N} = \{0, 1, 2, \dots\}$,

$$\sum_{i=0}^{n} 2 \cdot 3^i = 3^{n+1} - 1.$$

Induction for Divisibility and Inequalities

3. Prove by induction that $4 \mid 5^n + 3$ for every $n \in \mathbb{N} = \{0, 1, 2, \dots\}$.

4. Prove by induction that $3^n \geq 2n + 1$ for every $n \in \mathbb{N} = \{0, 1, 2, \dots\}$. State clearly where your inductive step uses an inequality that needs $k \geq 0$.

Strong Induction

5. A café sells muffins only in boxes of 3 and boxes of 7. An order of n muffins can be filled exactly when $n = 3a + 7b$ for some $a, b \in \mathbb{N} = \{0, 1, 2, \dots\}$.

a) Show that orders of 12, 13 and 14 muffins can be filled, and that an order of 11 cannot.

b) Use strong induction to prove that every order of $n \geq 12$ muffins can be filled.

c) Why does your proof need three base cases rather than one?

6. A sequence is defined by $d_1 = 1$, $d_2 = 2$, $d_3 = 3$ and $d_n = d_{n-1} + d_{n-3}$ for $n \geq 4$.

a) Compute d_4 through d_8.

b) Use strong induction to prove that $d_n \leq 2^{n-1}$ for every $n \in \mathbb{Z}^+$.

Recursively Defined Sequences

7.

a) Let $a_0 = 3$ and $a_n = 2a_{n-1} - 1$ for $n \geq 1$. Compute a_1 through a_5.

b) Let $b_1 = 1$, $b_2 = 4$ and $b_n = b_{n-1} + n\,b_{n-2}$ for $n \geq 3$. Compute b_3, b_4 and b_5.

c) Give a recursive definition (initial term and rule) of the sequence $5, 8, 11, 14, \ldots$ and of the sequence $2, 6, 18, 54, \ldots$, starting each at index 1.

d) For $n \in \mathbb{Z}^+$ let $s_n = \displaystyle\sum_{i=1}^{n} i^2$. Give a recursive definition of s_n that uses no $\sum$ sign.

8. A sequence is defined by $u_0 = 3$ and $u_n = u_{n-1}^2 - 2u_{n-1} + 2$ for $n \geq 1$.

a) Compute u_1, u_2 and u_3.

b) Prove by induction that $u_n = 2^{2^n} + 1$ for every $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$.

Recursive Algorithms and Their Correctness

9. The procedure CUBE takes $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ and is defined by: if $n = 0$, return 0; otherwise return CUBE$(n-1) + 3n(n-1) + 1$.

a) Trace CUBE(3): list the calls it makes and the value each one returns.

b) Explain why CUBE(n) terminates for every $n \in \mathbb{N}$.

c) Prove by induction that CUBE(n) returns n^3 for every $n \in \mathbb{N}$.

Synthesis — drawing on several topics in this unit

10. A sequence is defined by $w_0 = 2$ and $w_n = w_{n-1}^2 - w_{n-1} + 1$ for $n \geq 1$.

a) Compute w_1 through w_4.

b) Prove by induction that w_n leaves remainder 3 on division by 4 for every $n \in \mathbb{Z}^+$.

c) Why must the base case be $n = 1$ and not $n = 0$?