UNIVERSITY DISCRETE MATHEMATICS · PRACTICE WORKSHEET · TUTORINMONTREAL.CA

# Counting Techniques

Name: ______________________________ Date: ______________

## Concepts

- The Sum and Product Rules
- The Pigeonhole Principle
- Permutations (theory)
- Combinations
- The Binomial Theorem and Pascal's Identity
- Permutations and Combinations with Repetition (theory)

### The Sum and Product Rules

**1.** Let $A = \{1, 2, 3, 4\}$. A *relation* on $A$ is any subset of $A \times A$.

**a)** How many relations on $A$ are there?

**b)** How many of them are *reflexive*, that is, contain $(a, a)$ for every $a \in A$?

**c)** How many of them are *symmetric*, that is, contain $(b, a)$ whenever they contain $(a, b)$?

**d)** How many subsets of $\{1, 2, \ldots, 10\}$ contain 1 but do not contain 2?

**2.**

**a)** Given $720 = 2^4 \cdot 3^2 \cdot 5$, how many positive divisors does 720 have?

**b)** How many of those divisors are even?

**c)** How many positive integers less than 1000 have every digit odd?

### The Pigeonhole Principle

**3.**

**a)** What is the least number of integers that must be chosen from $\{1, 2, \ldots, 30\}$ to guarantee that two of them leave the same remainder on division by 7? Show that one fewer is not enough.

**b)** 58 students each choose one of 5 project topics. What is the largest number of students that is *guaranteed* to share a topic, whatever they choose?

**c)** How many students would there have to be to guarantee that at least 15 of them share a topic?

**4.** A set $S$ of 7 integers is chosen from $\{1, 2, \ldots, 12\}$.

**a)** Explain, naming the pigeons and the boxes, why $S$ must contain two consecutive integers.

**b)** Give a set of 6 integers from $\{1, 2, \ldots, 12\}$ with no two consecutive, and say what this shows about part (a).

### Permutations

**5.**

**a)** How many permutations of $\{1, 2, \ldots, 8\}$ are there?

**b)** In how many of them do the four even numbers occupy the first four positions?

**c)** In how many of them is 1 immediately followed by 2?

**d)** How many injective (one-to-one) functions $f\colon \{1, 2, 3\} \to \{1, 2, \ldots, 8\}$ are there?

### Combinations

**6.** A *bit string* is a string of 0s and 1s.

**a)** How many bit strings of length 12 have exactly five 1s?

**b)** How many bit strings of length 12 have at least ten 1s?

**c)** How many 4-element subsets of $\{1, 2, \ldots, 12\}$ have largest element 9?

**7.** A lattice path from $(0,0)$ to $(7,4)$ is a sequence of unit steps, each one to the right $(+1,0)$ or up $(0,+1)$.

**a)** How many lattice paths go from $(0,0)$ to $(7,4)$?

**b)** How many of them pass through the point $(3,2)$?

**c)** How many of them avoid $(3,2)$?

### The Binomial Theorem and Pascal's Identity

**8.**

**a)** Find the coefficient of $x^5y^3$ in the expansion of $(2x-y)^8$.

**b)** Find the term independent of $x$ in the expansion of $\left(x^2+\frac{3}{x}\right)^6$.

**c)** Find the coefficient of $x^4$ in $(1+x)^6(1-x)^2$.

**9.**

**a)** Given $\binom{15}{6} = 5005$ and $\binom{15}{7} = 6435$, find $\binom{16}{7}$ and $\binom{16}{9}$ without computing any factorial.

**b)** Find the integer $n \geq 5$ with $\binom{n}{3} = \binom{n}{5}$.

**c)** Evaluate $\sum_{k=0}^{8} (-1)^k \binom{8}{k} 3^{8-k}$.

### Permutations and Combinations with Repetition

**10.**

**a)** How many solutions does $x_1 + x_2 + x_3 + x_4 = 15$ have in nonnegative integers?

**b)** How many of those solutions have every $x_i \geq 2$?

**c)** How many different strings can be formed by rearranging all the letters $a, a, b, b, b, c, c, c$?

**Synthesis — drawing on several topics in this unit**

**11.** Count the functions $f\colon \{1,2,3,4\} \to \{1,2,3,4,5,6\}$ that are

**a)** arbitrary;

**b)** injective;

**c)** strictly increasing, $f(1) < f(2) < f(3) < f(4)$;

**d)** nondecreasing, $f(1) \le f(2) \le f(3) \le f(4)$.