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Recurrence Relations and Advanced Counting

Name: ______________________________ Date: ______________

Concepts

- Modelling a Counting Problem with a Recurrence
- Linear Homogeneous Recurrences with Distinct Roots
- Linear Homogeneous Recurrences with a Repeated Root (theory)
- The Inclusion-Exclusion Principle
- Counting Derangements and Onto Functions (theory)

Modelling a Counting Problem with a Recurrence

1. A door code is a string of n digits taken from $1, 2, \ldots, 9$ (no zero), and a code is *valid* if no two consecutive digits are both even. Let c_n be the number of valid codes of length n.

a) Find c_1 and c_2.

b) By considering the last digit of a valid code, show that $c_n = 5c_{n-1} + 20c_{n-2}$ for $n \geq 3$.

c) Compute c_4.

2. A librarian places n different books on three different shelves. On each shelf the books stand in a row from left to right, and a shelf may be left empty. Two placements are the same only when every shelf holds the same books in the same order. Let s_n be the number of placements.

a) Find s_1 and s_2.

b) Place the books in the order $1, 2, \ldots, n$. By counting the positions available to the last book once the first $n-1$ are on the shelves, find a recurrence expressing s_n in terms of s_{n-1} for $n \geq 2$, and justify it.

c) Compute s_5, and write s_n in terms of factorials.

Linear Homogeneous Recurrences with Distinct Roots

3. Solve the recurrence $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, with $a_0 = 3$ and $a_1 = 4$. Check your formula against a_2 computed from the recurrence.

4. A sequence satisfies $a_{n+1} = 2a_n + 8a_{n-1}$ for $n \geq 2$, with $a_1 = 4$ and $a_2 = 4$. Find a closed form for a_n, $n \geq 1$, and check it against a_3.

Linear Homogeneous Recurrences with a Repeated Root

5. Solve $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, with $a_0 = 2$ and $a_1 = 3$, and check your formula against a_2 and a_3 computed from the recurrence.

The Inclusion-Exclusion Principle

6. How many integers from 1 to 1000 inclusive are divisible by none of 4, 6 and 15?

7. A food bank has 14 identical one-hour shifts to hand out among 4 volunteers, and no volunteer may take more than 5 shifts (a volunteer may take none). In how many ways can the shifts be handed out? You may use the fact that the equation $x_1 + x_2 + \cdots + x_k = m$ has $\binom{m+k-1}{k-1}$ solutions in nonnegative integers.

Counting Derangements and Onto Functions

8.

a) Five distinct manuscripts are sent to three editors, every manuscript going to exactly one editor, and every editor must receive at least one. In other words, count the surjective (*onto*) functions from a 5-element set to a 3-element set: the functions that hit every element of the target.

b) Five reviewers each wrote a report, and the reports are collected and handed back so that each reviewer checks one report — never their own. In how many ways can this be done?

Synthesis — drawing on several topics in this unit

9. A signal occupies n consecutive time slots and is built as a sequence of pulses: a short pulse fills one slot and can be sent at any of 2 frequencies; a long pulse fills two slots and can be sent at any of 3 frequencies. Let a_n be the number of different signals filling exactly n slots, with $a_0 = 1$ (the empty signal).

a) Explain why $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$, and give a_1.

b) Solve the recurrence, and check your formula against a_4.