University Discrete Mathematics · Practice Worksheet · tutorinmontreal.ca

Graphs and Trees

Name: ______________________ Date: ____________

Concepts

- Graph Terminology and Special Graphs
- The Handshake Theorem and Degree Sequences
- Adjacency Matrices and Graph Isomorphism (theory)
- Paths, Cycles and Connectivity
- Euler and Hamilton Paths and Circuits
- Shortest Paths with Dijkstra's Algorithm
- Trees and Their Properties (theory)
- Planar Graphs and Euler's Formula (theory)

Graph Terminology and Special Graphs

1. Throughout this worksheet, *graph* means a *simple* undirected graph $G = (V, E)$: no loops, and no two edges joining the same pair of vertices. The edge $\{x, y\}$ is written xy.

Let $V = \{a, b, c, d, e, f, g, h\}$ and $E = \{ab,\ ac,\ bc,\ cd,\ de,\ df,\ ef,\ fg\}$.

a) Give $\deg(v)$ for every vertex. Name the isolated vertex (degree 0) and the pendant vertex (degree 1).

b) List the neighbours of d. How many edges does the subgraph induced by $\{c, d, e, f\}$ have (every edge of G with both ends in that set)?

c) The complement $\overline{G}$ has the same vertices, two distinct vertices being adjacent in $\overline{G}$ exactly when they are not adjacent in G. How many edges does $\overline{G}$ have?

2. A graph is *bipartite* when its vertices can be split into two sets so that every edge joins a vertex of one set to a vertex of the other. For each graph below, give the number of vertices, the number of edges, and say whether it is bipartite, with a reason (a valid split, or why none exists).

a) K_6, the complete graph on 6 vertices.

b) C_7, the cycle on 7 vertices.

c) $K_{3,5}$, the complete bipartite graph with parts of sizes 3 and 5.

d) P_6, the path $v_1 v_2 \cdots v_6$ on 6 vertices.

The Handshake Theorem and Degree Sequences

3. A graph has 8 vertices and 11 edges. Seven of its vertices have degrees $1, 1, 2, 2, 3, 3, 4$.

a) Find the degree of the eighth vertex, naming the theorem you use.

b) Explain why no graph has degrees $5, 4, 3, 3, 2, 1, 1$.

4. Decide whether each list is the degree sequence of a graph. If it is, give such a graph by listing its edges; if it is not, explain why.

a) $3, 3, 3, 3, 2, 2$

b) $4, 4, 4, 1, 1$

c) $5, 3, 2, 2, 2$

Adjacency Matrices and Graph Isomorphism

5. The graph G on vertices $v_1, \ldots, v_5$ has adjacency matrix A, where $a_{ij} = 1$ if v_iv_j is an edge and $a_{ij} = 0$ otherwise:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

a) List the edges of G and the degree of each vertex.

b) The (i, j) entry of A^2 is $(A^2)_{ij} = \sum_{k=1}^{5} a_{ik}a_{kj}$, the number of walks of length 2 from v_i to v_j. Compute $(A^2)_{14}$ and list the walks it counts.

c) Explain why $(A^2)_{ii} = \deg(v_i)$ for every i.

Paths, Cycles and Connectivity

6. Let G have vertex set $\{1, 2, \ldots, 9\}$ and edges $12,\ 23,\ 13,\ 34,\ 56,\ 67,\ 78,\ 58$.

a) Give the connected components of G as sets of vertices. Is G connected?

b) Give a cycle of length 4 in G.

c) Find the distance $d(1, 4)$ (the length of a shortest path) and a path that realises it.

d) Give a walk of length 4 from 2 to 4 that is not a path.

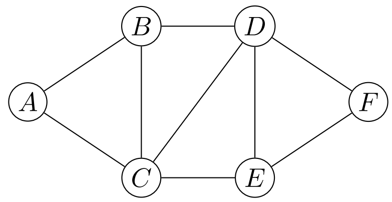
7. A *bridge* of a graph is an edge whose deletion increases the number of connected components; a *cut vertex* is a vertex whose deletion, together with its edges, does so. The connected graph G has vertices $a, \ldots, h$ and edges

$$ab,\ bc,\ ca,\ cd,\ de,\ ef,\ fg,\ gd,\ fh.$$

Find every bridge and every cut vertex, with a reason for each.

Euler and Hamilton Paths and Circuits

8. The graph G has vertices A, B, C, D, E, F and the nine edges AB, AC, BC, BD, CD, CE, DE, DF, EF, drawn below: A on the left, F on the right, B and D along the top, C and E along the bottom, with a diagonal edge from C to D.



a) Give the degree of every vertex.

b) Does G have an Euler circuit? An Euler path? Justify with the degrees, and write one down if it exists.

9.

a) The graph G has vertices $1, \dots, 7$ and edges 12, 14, 15, 24, 26, 34, 36, 37, 56, 57. Find a Hamilton circuit (a circuit visiting every vertex exactly once).

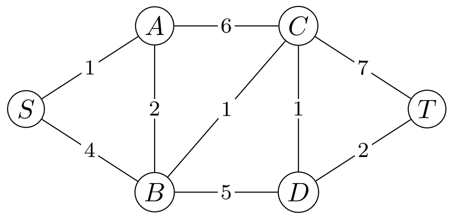
b) The graph H has vertices a, b, c, d, e and edges ab, bc, ca, cd, de, ec (two triangles sharing the vertex c). Show that H has an Euler circuit but no Hamilton circuit.

Shortest Paths with Dijkstra's Algorithm

10. A courier company has six depots S, A, B, C, D, T. The travel times, in minutes, along the direct roads are

$$SA = 1,\ SB = 4,\ AB = 2,\ AC = 6,\ BC = 1,\ BD = 5,\ CD = 1,\ CT = 7,\ DT = 2,$$

as drawn below (S on the left, T on the right, A and C along the top, B and D along the bottom, with a diagonal road from B to C).



Run Dijkstra's algorithm from S. Give the order in which the depots receive permanent labels, with their labels, then the fastest route from S to T and its time.

11. A cycling network joins six stations P, Q, R, U, V, W by trails with these lengths, in kilometres:

$$PQ = 7,\ PR = 3,\ QR = 2,\ QU = 4,\ RU = 8,\ RV = 9,\ UV = 1,\ UW = 3,\ VW = 5.$$

Use Dijkstra's algorithm from P to find the shortest distance from P to every other station and the station just before it on a shortest route. Then give the shortest route from P to W.

Trees and Their Properties

12. A *tree* is a connected graph with no cycle, and a *leaf* is a vertex of degree 1. A tree has exactly two vertices of degree 4, three of degree 3, one of degree 2, and every other vertex is a leaf. How many leaves does it have, and how many vertices?

Planar Graphs and Euler's Formula

13. Euler's formula: a connected graph drawn in the plane without crossings, with v vertices, e edges and f faces (the unbounded outer face included), satisfies $v - e + f = 2$.

a) A connected planar graph has 10 vertices, each of degree 3. How many faces does any crossing-free drawing of it have?

b) A connected graph drawn without crossings has 12 edges and 8 faces. How many vertices does it have?

Synthesis — drawing on several topics in this unit

14. The graph K has vertices $1, \dots, 6$, grouped into the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$. Two distinct vertices are adjacent exactly when they are *not* in the same pair.

a) Give the degree of every vertex and use the handshake theorem to find the number of edges.

b) Does K have an Euler circuit? If so, give one.

c) Give a Hamilton circuit of K.

d) K can be drawn without crossings. How many faces does such a drawing have?

Want the harder problems and the worked solutions?

The *Challenge Problems* set and the *Answer Keys* for Graphs and Trees — with the reasoning written out in full — are available at **tutorinmontreal.ca**.