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Vector Spaces and Subspaces

Name: ____ Date: __________________

Concepts

- Vectors in n-Space and Linear Combinations
- The Vector Space Axioms
- Consequences of the Axioms (theory)
- Spaces of Polynomials, Matrices and Functions
- The Subspace Test
- Solution Sets of Homogeneous Systems as Subspaces (theory)

Vectors in n-Space and Linear Combinations

1. In $\mathbb{R}^5$, let $\vec{u} = (1, -2, 0, 3, 1)$, $\vec{v} = (2, 1, -1, 0, 4)$ and $\vec{w} = (0, 3, 2, -1, -2)$.

a) Compute the linear combination $2\vec{u} - 3\vec{v} + \vec{w}$.

b) Solve $3(\vec{x} - \vec{u}) = \vec{x} + 2\vec{v}$ for the vector $\vec{x} \in \mathbb{R}^5$.

c) Is $\vec{v}$ a scalar multiple of $\vec{u}$? Justify in one line.

2. Consider the linear system

$$\begin{aligned} x_1 \qquad + 2x_3 &= 7 \\ 2x_1 + x_2 \qquad &= 5 \\ 3x_2 + x_3 &= -1 \\ -x_1 + x_2 + x_3 &= -2 \end{aligned}$$

a) Rewrite it as a single vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$ in $\mathbb{R}^4$, naming the four vectors.

b) Find the weights x_1, x_2, x_3, and check them by computing the linear combination.

c) Without solving anything further, write $(14, 10, -2, -4)$ as a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

The Vector Space Axioms

3. On P_2, the polynomials of degree at most 2, keep the usual addition of polynomials but define a new scalar multiplication that discards the x^2 term:

$$c \odot \left(a_0 + a_1x + a_2x^2\right) = ca_0 + ca_1x.$$

The addition axioms all hold, since addition is unchanged. Check the four scalar-multiplication axioms,

$$c \odot (p+q) = c \odot p + c \odot q, \quad (c+d) \odot p = c \odot p + d \odot p, \quad c \odot (d \odot p) = (cd) \odot p, \quad 1 \odot p = p,$$

and show that exactly one of them fails, with a specific counterexample.

4. On $\mathbb{R}^2$, keep the usual scalar multiplication $c(x, y) = (cx, cy)$ but define a new addition

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 + 1,\ y_1 + y_2 - 2).$$

a) Find the vector $\vec{z}$ with $\vec{u} \oplus \vec{z} = \vec{u}$ for every $\vec{u}$ (the zero vector for $\oplus$).

b) Find the negative of $(3, -2)$ for $\oplus$, that is, the vector $\vec{n}$ with $(3, -2) \oplus \vec{n} = \vec{z}$.

c) Show that the axiom $(c + d)\vec{u} = c\vec{u} \oplus d\vec{u}$ fails, with a specific counterexample.

Consequences of the Axioms

5. Here is a proof that $0\vec{v} = \vec{0}$ for every vector $\vec{v}$ in any vector space V. Here 0 is the real number and $\vec{0}$ the zero vector of V. For each numbered step, give its justification in words: the property of real numbers, or the vector space axiom, that it uses.

$$\begin{aligned} 0\vec{v} &= (0+0)\vec{v} \quad (1) \\ &= 0\vec{v} + 0\vec{v} \quad (2) \end{aligned}$$

Add the vector $-(0\vec{v})$ to both sides of $0\vec{v} = 0\vec{v} + 0\vec{v}$:

$$\begin{aligned} 0\vec{v} + \big(-(0\vec{v})\big) &= \big(0\vec{v} + 0\vec{v}\big) + \big(-(0\vec{v})\big) && (3) \\ \vec{0} &= 0\vec{v} + \Big(0\vec{v} + \big(-(0\vec{v})\big)\Big) && (4) \\ \vec{0} &= 0\vec{v} + \vec{0} && (5) \\ \vec{0} &= 0\vec{v} && (6) \end{aligned}$$

Spaces of Polynomials, Matrices and Functions

6. Operations in three spaces that are not $\mathbb{R}^n$, each with its usual addition and scalar multiplication.

a) In P_3, let $p(x) = 2 - x + 3x^3$ and $q(x) = 1 + 4x - x^2 + x^3$. Compute $3p - 2q$.

b) In $M_{2\times3}$, let $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 & 1 \\ -2 & 2 & 3 \end{bmatrix}$. Compute $2A - B$, and write down the zero vector of $M_{2\times3}$ and the negative of A.

c) In the space $F(\mathbb{R})$ of all functions $\mathbb{R} \to \mathbb{R}$, let $f(x) = e^x$ and $g(x) = x^2$. Write down the function $3f - 2g$ and its value at $x = 0$. What is the zero vector of $F(\mathbb{R})$?

7. In $M_{2\times2}$, let

$$E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad H = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$

a) Find real numbers a, b, c with $aE + bF + cG = H$, writing down the four equations that the entries give.

b) Explain in one sentence why this is the same computation as writing a vector of $\mathbb{R}^4$ as a linear combination of three others.

The Subspace Test

8. Let $W = \{p \in P_3 \mid p(1) = p(-1)\}$.

a) Use the subspace test to show that W is a subspace of P_3.

b) Writing $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, turn the condition $p(1) = p(-1)$ into a condition on the coefficients, and decide whether $4 - 3x + x^2 + 3x^3$ and $1 + x + x^3$ lie in W.

9. Each set below carries the usual operations. Decide whether it is a subspace. If it is, verify the three conditions; if it is not, name one condition that fails and give a specific counterexample.

a) $W_1 = \{\vec{x} \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 1\}$

b) $W_2 = \{\vec{x} \in \mathbb{R}^4 \mid x_1 x_4 = 0\}$

c) $W_3 = \{A \in M_{2\times 2} \mid A^T = A\}$

d) $W_4 = \{\vec{x} \in \mathbb{R}^4 \mid x_2 \geq |x_1|\}$

Solution Sets of Homogeneous Systems as Subspaces

10. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 4 & 1 & 1 \\ -1 & -2 & 1 & 4 \end{bmatrix}, \qquad W = \{\vec{x} \in \mathbb{R}^4 \mid A\vec{x} = \vec{0}\}.$$

a) Verify that $\vec{u} = (-2, 1, 0, 0)$ and $\vec{v} = (1, 0, -3, 1)$ lie in W.

b) Without solving the system, explain why $3\vec{u} - 2\vec{v}$ lies in W, and write it down.

c) The system $A\vec{x} = (1, 2, -1)$ has the solution $(1, 0, 0, 0)$. Is its solution set a subspace of $\mathbb{R}^4$? Justify.

Synthesis — drawing on several topics in this unit

11. Let $W = \{p \in P_3 \mid p(1) = 0 \text{ and } p'(1) = 0\}$, where p' is the derivative of p.

a) Use the subspace test to show that W is a subspace of P_3.

b) Writing $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, turn the two conditions into a homogeneous linear system in a_0, a_1, a_2, a_3 and solve it.

c) Deduce that every $p \in W$ is a linear combination of $q_1(x) = 1 - 2x + x^2$ and $q_2(x) = 2 - 3x + x^3$.

d) Decide whether $7 - 12x + 3x^2 + 2x^3$ and $x^3 - 1$ lie in W; for any that does, give its weights on q_1 and q_2.