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Span and Linear Independence

Name: ________________________ Date: ______________

Concepts

- The Span of a Set of Vectors
- Testing Membership in a Span
- Linear Independence in n-Space
- Linear Independence of Polynomials, Matrices and Functions
- Dependence Relations and Redundant Vectors (theory)

The Span of a Set of Vectors

1. In $\mathbb{R}^4$ let $W = \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$, where $\vec{v}_1 = (1, 0, 2, -1)$ and $\vec{v}_2 = (0, 1, -1, 3)$.

a) Find a system of two linear equations in x_1, x_2, x_3, x_4 whose solution set is exactly W.

b) Use your equations to decide which of $(3, -1, 7, -6)$ and $(1, 1, 1, 1)$ lies in W.

2. In P_2, the space of polynomials of degree at most 2 with real coefficients, let $W = \operatorname{span}\{1+x,\ x+x^2\}$.

a) Show that every $p \in W$ satisfies $p(-1) = 0$.

b) Show conversely that every $p \in P_2$ with $p(-1) = 0$ lies in W, by writing such a p explicitly as a linear combination of $1 + x$ and $x + x^2$.

c) Conclude that $W = \{p \in P_2 \mid p(-1) = 0\}$, and give one polynomial of P_2 that is not in W.

Testing Membership in a Span

3. In $\mathbb{R}^4$ let $\vec{u}_1 = (1, -1, 0, 2)$, $\vec{u}_2 = (0, 1, 2, -1)$ and $\vec{u}_3 = (2, 0, 3, 1)$.

a) Decide whether $\vec{b} = (4, -3, 1, 6)$ lies in $\operatorname{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. If it does, give the weights.

b) Decide whether $\vec{e}_1 = (1, 0, 0, 0)$ lies in the same span.

4. In P_3, the space of polynomials of degree at most 3, let $p_1 = 1+x^3$, $p_2 = x-x^2$ and $p_3 = 1+x+x^2+x^3$.

a) Show that $r = 3 + x + 3x^2 + 3x^3$ is a linear combination of p_1, p_2, p_3, and give the weights.

b) Show that $q = 2 + x + 3x^2 + 5x^3$ is not.

Linear Independence in n-Space

5. Decide whether each set in $\mathbb{R}^4$ is linearly independent. For a dependent set, give an explicit dependence relation.

a) $\{(1, 2, -1, 0),\ (0, 1, 3, 1),\ (1, 4, 5, 2)\}$

b) $\{(1, 1, 0, 2),\ (0, 1, 1, 1),\ (2, 0, -1, 3),\ (1, 0, 1, 0)\}$

6. Decide, without row reducing anything, whether each set is linearly independent, and give the reason in one sentence.

a) $\{(1,0,2,1),\ (0,3,1,-1),\ (2,2,0,5),\ (1,1,1,1),\ (4,0,-3,2)\}$ in $\mathbb{R}^4$

b) $\{(3,1,0,-2,5),\ (0,0,0,0,0),\ (1,4,-1,0,2)\}$ in $\mathbb{R}^5$

c) $\{(2,-4,6,0,8),\ (-3,6,-9,0,-12)\}$ in $\mathbb{R}^5$

d) $\{(1,0,0,0,5),\ (0,0,3,0,0)\}$ in $\mathbb{R}^5$

Linear Independence of Polynomials, Matrices and Functions

7. Decide whether each set is linearly independent. For a dependent set, give an explicit dependence relation.

a) In P_3 (polynomials of degree at most 3): $\{1+x+x^2,\ x+x^2+x^3,\ 1-x^3\}$.

b) In $M_{2\times 2}$ (real 2×2 matrices): $\left\{\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 3 & 1\end{bmatrix}, \begin{bmatrix}2 & 1\\ -1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\right\}$.

8. Here functions $\mathbb{R} \to \mathbb{R}$ are vectors, and a set $\{f_1, \dots, f_k\}$ is linearly dependent when there are real numbers $c_1, \dots, c_k$, not all zero, with $c_1 f_1(x) + \cdots + c_k f_k(x) = 0$ *for every real* x.

a) Show that $\{\sin\left(x + \frac{\pi}{3}\right),\ \sin x,\ \cos x\}$ is linearly dependent by giving an explicit relation. You may use $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

b) Show that $\{1,\ x,\ e^x\}$ is linearly independent, by evaluating a vanishing linear combination at $x = 0$, $x = 1$ and $x = 2$.

Dependence Relations and Redundant Vectors

9. In $\mathbb{R}^4$ consider the list

$$\vec{v}_1 = (1,2,0,1), \quad \vec{v}_2 = (2,4,0,2), \quad \vec{v}_3 = (0,1,1,-1), \quad \vec{v}_4 = (1,3,1,0), \quad \vec{v}_5 = (0,0,1,1).$$

a) Find every vector in the list that is a linear combination of the vectors *before* it, and write each such combination explicitly.

b) Name three vectors of the list whose span is $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_5\}$, and say why nothing is lost.

Synthesis — drawing on several topics in this unit

10. In $M_{2\times 2}$, the space of real 2×2 matrices, let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad W = \operatorname{span}\{A_1, A_2, A_3\}.$$

a) Show that $\{A_1, A_2, A_3\}$ is linearly dependent, and give a dependence relation.

b) Show that $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, d = a,\ c = a + b \right\}$.

c) Decide whether $B = \begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix}$ and I_2 lie in W; for any that does, write it as a combination of A_1 and A_2.