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Basis, Dimension and Coordinates

Name: ______________________________ Date: ______________

Concepts

- Bases and the Standard Bases
- The Dimension of a Vector Space
- Reducing a Spanning Set and Extending an Independent Set (theory)
- Coordinate Vectors Relative to a Basis
- Coordinates in Polynomial and Matrix Spaces (theory)

Bases and the Standard Bases

1. Decide whether each set is a basis of $\mathbb{R}^4$. Where it is not, write down an explicit dependence relation among its vectors.

a) $\vec{v}_1 = (1, 0, 2, 1)$, $\vec{v}_2 = (0, 1, 1, -1)$, $\vec{v}_3 = (1, 1, 0, 2)$, $\vec{v}_4 = (2, 1, 3, 1)$

b) $\vec{w}_1 = (1, -1, 2, 0)$, $\vec{w}_2 = (0, 1, 1, 3)$, $\vec{w}_3 = (2, -3, 3, -3)$, $\vec{w}_4 = (1, 1, 0, 1)$

2. Write down the standard basis of $M_{2\times 2}$. Then show, directly from the definition of a basis, that

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

is also a basis of $M_{2\times 2}$.

The Dimension of a Vector Space

3. Find a basis and the dimension of each subspace.

a) $W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_3 = x_1 + x_2 \text{ and } x_4 = 2x_5\}$

b) $U = \{A \in M_{2\times 3} \mid \text{each row of } A \text{ has entries summing to } 0\}$

4. Answer each part without solving any system.

a) State $\dim P_4$ and $\dim M_{3\times 2}$.

b) Explain why any six polynomials in P_4 are linearly dependent.

c) Can five matrices span $M_{3\times 2}$? Explain.

d) Someone has checked that five particular polynomials in P_4 are linearly independent. Explain why they must form a basis of P_4.

Reducing a Spanning Set and Extending an Independent Set

5. The polynomials $p_1(x) = 1 + x^2$ and $p_2(x) = x - x^3$ are linearly independent in P_3. Extend $\{p_1, p_2\}$ to a basis of P_3 by adding polynomials from the standard basis $\{1, x, x^2, x^3\}$, and justify that the result is a basis.

Coordinate Vectors Relative to a Basis

6. The vectors $\vec{b}_1 = (1, 1, 0, 0)$, $\vec{b}_2 = (0, 1, 1, 0)$, $\vec{b}_3 = (0, 0, 1, 1)$, $\vec{b}_4 = (0, 0, 0, 1)$ form a basis $\mathcal{B}$ of $\mathbb{R}^4$. Find the coordinate vector $[\vec{x}]_{\mathcal{B}}$ of $\vec{x} = (3, 1, 4, 2)$.

7. The vectors $\vec{b}_1 = (1, 0, 1, 0)$, $\vec{b}_2 = (0, 1, 0, 1)$, $\vec{b}_3 = (1, -1, 0, 0)$, $\vec{b}_4 = (0, 0, 1, 1)$ form a basis $\mathcal{B}$ of $\mathbb{R}^4$.

a) Find $\vec{v}$ if $[\vec{v}]_{\mathcal{B}} = (2, -1, 3, 1)$.

b) A vector $\vec{u}$ has $[\vec{u}]_{\mathcal{B}} = (1, 0, -2, 4)$. Find $[3\vec{v} - 2\vec{u}]_{\mathcal{B}}$ without finding $\vec{u}$ or solving any system.

Coordinates in Polynomial and Matrix Spaces

8. Find each coordinate vector.

a) $[p]_{\mathcal{B}}$, where $p(x) = 2x^2 - 5x + 4$ and $\mathcal{B} = \{1,\ x - 3,\ (x - 3)^2\}$ is a basis of P_2.

b) $[A]_{\mathcal{C}}$, where $A = \begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}$ and

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

is a basis of $M_{2\times 2}$.

Synthesis — drawing on several topics in this unit

9. Let $W = \{p \in P_3 \mid p(0) = p(2)\}$, a subspace of P_3, and let $\mathcal{B} = \{1,\ x^2 - 2x,\ x^3 - 4x\}$.

a) Show that $\mathcal{B}$ is a basis of W, and state $\dim W$.

b) Show that $q(x) = x^3 - 3x^2 + 2x + 5$ lies in W, and find $[q]_{\mathcal{B}}$.