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Rank and the Fundamental Subspaces

Name: ______________________________ Date: ______________

Concepts

- Solution Sets in Parametric Vector Form (theory)
- The Null Space of a Matrix
- The Column Space and the Row Space
- Bases for the Null, Column and Row Spaces
- Rank, Nullity and the Rank Theorem
- Rank and the Consistency of a Linear System (theory)

Solution Sets in Parametric Vector Form

1. The augmented matrix of a linear system in the unknowns x_1, x_2, x_3, x_4, x_5 has been row reduced to

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

a) Write the solution set of the system in parametric vector form, $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$.

b) Write, without further work, the solution set of the homogeneous system with the same coefficient matrix.

The Null Space of a Matrix

2. Let $A = \begin{bmatrix} 1 & -1 & 2 & 0 & 3 \\ 2 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 & 4 \end{bmatrix}$.

a) $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^k$. What is k?

b) Determine whether $\vec{u} = (3, -7, -5, 1, 0)$ and $\vec{w} = (1, 1, 1, 1, -1)$ belong to $\mathrm{Nul}(A)$.

3. Let $B = \begin{bmatrix} 1 & 2 & -1 & 0 & 3 \\ 2 & 4 & 0 & 2 & 2 \\ -1 & -2 & 3 & 2 & -7 \end{bmatrix}$. Describe $\mathrm{Nul}(B)$ as the span of a list of vectors, one for each free variable.

The Column Space and the Row Space

4. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ -1 & 1 & -4 \\ 0 & 2 & -1 \end{bmatrix}$. For each of $\vec{b} = (8, 12, -15, -5)$ and $\vec{c} = (3, 5, -4, -1)$, decide whether it lies in $\operatorname{Col}(A)$. If it does, write it as a linear combination of the columns of A.

5. Let $A = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 & -2 \\ 1 & 1 & 4 & 0 & 0 \end{bmatrix}$, with rows $\vec{r}_1, \vec{r}_2, \vec{r}_3$.

a) $\operatorname{Row}(A)$ and $\operatorname{Col}(A)$ are subspaces of which $\mathbb{R}^k$?

b) Show that $\operatorname{Col}(A) = \mathbb{R}^3$.

c) Decide whether $(2, -1, 3, -3, 4)$ and $(1, 1, 1, 1, 1)$ lie in $\operatorname{Row}(A)$.

Bases for the Null, Column and Row Spaces

6. The matrix A below is row equivalent to the matrix R beside it.

$$A = \begin{bmatrix} 1 & -3 & 0 & 1 & 3 \\ 2 & -6 & 1 & 4 & 5 \\ -1 & 3 & 2 & 3 & -7 \\ 3 & -9 & 1 & 5 & 8 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find a basis for each of $\mathrm{Nul}(A)$, $\mathrm{Col}(A)$ and $\mathrm{Row}(A)$.

7. Let W be the set of all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ satisfying both

$$x_1 - 2x_2 + x_3 - x_4 + 3x_5 = 0 \qquad \text{and} \qquad x_2 + x_3 + 2x_4 - x_5 = 0.$$

Write W as the null space of a matrix, and use that to find a basis for W and its dimension.

Rank, Nullity and the Rank Theorem

8. A matrix A has size 6×9 and $\operatorname{rank}(A) = 4$. For each of $\operatorname{Nul}(A)$, $\operatorname{Col}(A)$, $\operatorname{Row}(A)$ and $\operatorname{Nul}(A^T)$, say which $\mathbb{R}^k$ it is a subspace of and give its dimension.

9. Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 5 & 1 & 1 \\ -1 & -1 & 2 & k \\ 1 & 1 & -2 & 1 \end{bmatrix}$, where k is a real number.

a) Find $\operatorname{rank}(A)$ for every value of k.

b) For the value of k at which the rank is smallest, find a basis of $\operatorname{Nul}(A)$ and confirm the rank theorem.

Rank and the Consistency of a Linear System

10. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \\ 0 & 1 & 2 \\ 3 & 7 & -1 \end{bmatrix}$ and $\vec{b} = (1, 3, h, g)$, where h and g are real numbers. Compare $\operatorname{rank}(A)$ with the rank of the augmented matrix $[\,A \mid \vec{b}\,]$ to find every pair (h, g) for which $A\vec{x} = \vec{b}$ is consistent. For those values, how many free parameters does the solution set have?

Synthesis — drawing on several topics in this unit

11. Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ -1 & 1 & -3 & 0 & 4 \\ 2 & 1 & 3 & 1 & 2 \\ 0 & 1 & -1 & -1 & 2 \end{bmatrix}, \qquad \vec{b} = (1, -1, 3, -1).$$

a) Find $\operatorname{rank}(A)$ and a basis of $\operatorname{Nul}(A)$, and check the rank theorem.

b) Verify that $\vec{p} = (1, 0, 0, 1, 0)$ solves $A\vec{x} = \vec{b}$, and write the full solution set in parametric vector form.

c) Is $A\vec{x} = \vec{c}$ consistent for every $\vec{c} \in \mathbb{R}^4$? If not, give one $\vec{c}$ for which it is not.