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Linear Transformations

Name: ______________________________ Date: ______________

Concepts

- Verifying That a Map Is Linear
- The Standard Matrix of a Linear Transformation
- Rotations, Reflections, Projections and Shears
- The Kernel and Range of a Linear Transformation
- The Dimension Theorem for Linear Transformations
- One-to-One and Onto Transformations (theory)
- Composition of Linear Transformations (theory)
- Invertible Transformations and Isomorphism (theory)

Verifying That a Map Is Linear

1. A map $T\colon V \to W$ between vector spaces is *linear* when $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$ and $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u}, \vec{v} \in V$ and every real scalar c. Decide whether each map is linear. For a linear map, verify both properties; for a map that is not, give specific vectors (and a scalar, if needed) at which a property fails.

a) $T\colon \mathbb{R}^4 \to \mathbb{R}^2$, $T(x_1, x_2, x_3, x_4) = (x_1 - 2x_3,\ x_2 + x_4)$

b) $S\colon \mathbb{R}^4 \to \mathbb{R}^2$, $S(x_1, x_2, x_3, x_4) = (x_1 + 1,\ x_3)$

c) $U\colon \mathbb{R}^4 \to \mathbb{R}^2$, $U(x_1, x_2, x_3, x_4) = (x_1x_2,\ x_4)$

2. Let P_2 be the vector space of polynomials of degree at most 2, and define $T\colon P_2 \to P_2$ by

$$T\big(p(x)\big) = p(x+1) - p(x).$$

a) Show that T is linear.

b) Compute $T(3 - x + 2x^2)$.

The Standard Matrix of a Linear Transformation

3. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation

$$T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 - x_4,\ 3x_2 + x_3,\ x_1 - x_3 + 4x_4).$$

a) Find the standard matrix A of T, so that $T(\vec{x}) = A\vec{x}$.

b) Use A to compute $T(1, -1, 2, 0)$.

4. A linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^4$ satisfies

$$T(1,1) = (3,1,0,2) \qquad \text{and} \qquad T(1,-1) = (1,-1,4,0).$$

Find the standard matrix of T.

Rotations, Reflections, Projections and Shears

5. Find the standard matrix of each linear operator on $\mathbb{R}^2$, then answer the question that follows it.

a) R, the counterclockwise rotation about the origin through $\frac{2\pi}{3}$. Find $R(2,\, 2\sqrt{3})$.

b) H, the shear parallel to the x-axis that sends $(0,1)$ to $(-2,1)$ and fixes every point of the x-axis. Find the image under H of the unit square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, and its area.

6. Let L be the line $y = 2x$ in $\mathbb{R}^2$.

a) Find the standard matrix P of the orthogonal projection onto L, and verify that $P^2 = P$.

b) Find the standard matrix F of the reflection in L, and verify that $F^2 = I_2$.

c) Find the projection and the reflection of $(5, 0)$.

The Kernel and Range of a Linear Transformation

7. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 4 & 1 & 1 \\ -1 & -2 & 1 & 4 \end{bmatrix}.$$

a) Find a basis for the kernel $\ker T = \{\vec{x} \in \mathbb{R}^4 \mid T(\vec{x}) = \vec{0}\}$.

b) Find a basis for the range $\operatorname{im} T = \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^4\}$, and describe it by a single linear equation in (y_1, y_2, y_3).

8. Let P_3 be the vector space of polynomials of degree at most 3, and let $T\colon P_3 \to \mathbb{R}^2$ be the linear map $T(p) = (p(0),\, p(2))$. Find a basis for $\ker T$ and a basis for $\operatorname{im} T$.

The Dimension Theorem for Linear Transformations

9. Let $M_{2\times 2}$ be the vector space of real 2×2 matrices, and let $T\colon M_{2\times 2} \to M_{2\times 2}$ be the linear map $T(A) = A - A^T$.

a) Describe $\ker T$ and find its dimension.

b) Describe $\operatorname{im} T$ and find its dimension.

c) Check that your answers agree with the dimension theorem.

10. Let $T\colon \mathbb{R}^7 \to \mathbb{R}^5$ be linear. Use the dimension theorem, $\dim\ker T + \dim\operatorname{im} T = 7$, to answer each part.

a) If T is onto $\mathbb{R}^5$, what is $\dim\ker T$?

b) If $\dim\ker T = 4$, what is $\dim\operatorname{im} T$, and is T onto?

c) What is the smallest possible value of $\dim\ker T$? Can T be one-to-one?

One-to-One and Onto Transformations

11. For each linear map, decide whether it is one-to-one and whether it is onto, and justify each verdict with the kernel or the dimension of the range.

a) $T\colon \mathbb{R}^3 \to \mathbb{R}^4$, $T(x_1, x_2, x_3) = (x_1,\ x_1 + x_2,\ x_2 + x_3,\ x_3)$.

b) $E\colon P_2 \to \mathbb{R}^3$, $E(p) = \big(p(0),\ p(1),\ p(2)\big)$, where P_2 is the space of polynomials of degree at most 2.

Composition of Linear Transformations

12. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ and $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear maps

$$T(x_1, x_2, x_3, x_4) = (x_1 + x_2,\ x_3 - x_4), \qquad S(y_1, y_2) = (y_1,\ y_1 + y_2,\ 2y_2).$$

a) Find the standard matrix of $S \circ T$, and state its domain and codomain.

b) Compute $(S \circ T)(1, 2, 3, 4)$ two ways: with the matrix, and by applying T and then S.

c) Is $T \circ S$ defined? Explain.

Invertible Transformations and Isomorphism

13. Let $M_{2\times 2}$ be the space of real 2×2 matrices and P_3 the space of polynomials of degree at most 3. Define $T\colon M_{2\times 2} \to P_3$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + (a + b)x + (b + c)x^2 + (c + d)x^3.$$

a) Show that T is invertible by finding a formula for T^{-1}.

b) Find the matrix A with $T(A) = 1 + 3x + 2x^2 - x^3$.

Synthesis — drawing on several topics in this unit

14. Let P_3 be the space of polynomials of degree at most 3, and define $T\colon P_3 \to \mathbb{R}^3$ by

$$T(p) = \big(p(1),\ p'(1),\ p''(1)\big).$$

a) Show that T is linear.

b) Find a basis for $\ker T$.

c) Use the dimension theorem to find $\dim \operatorname{im} T$. Is T onto? Is it one-to-one?