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Change of Basis and Matrix Representations

Name: ____   Date: __________________

Concepts

- The Change of Basis Matrix
- The Matrix of a Transformation Relative to Given Bases
- Matrix Representations on Polynomial and Matrix Spaces (theory)
- The Change of Basis Formula for a Linear Operator
- Similar Matrices and Similarity Invariants (theory)

The Change of Basis Matrix

1. In P_2, the space of polynomials of degree at most 2, let $\mathcal{S} = \{1,\, x,\, x^2\}$ be the standard basis and $\mathcal{B} = \{1,\ 1+x,\ (1+x)^2\}$. The change-of-basis matrix $P_{\mathcal{S}\leftarrow\mathcal{B}}$ is the matrix whose jth column is the $\mathcal{S}$-coordinate vector of the jth vector of $\mathcal{B}$.

a) Find $P_{\mathcal{S}\leftarrow\mathcal{B}}$ and $P_{\mathcal{B}\leftarrow\mathcal{S}}$.

b) Use a change-of-basis matrix to find $[p]_{\mathcal{B}}$ for $p(x) = 3 - 2x + 5x^2$.

c) Find the polynomial q with $[q]_{\mathcal{B}} = (2, -1, 3)$.

2. Let $W = \text{span}\{\vec{c}_1, \vec{c}_2\}$, a plane through the origin in $\mathbb{R}^4$, where

$$\vec{c}_1 = (1,0,1,2), \quad \vec{c}_2 = (0,1,-1,1), \qquad \vec{b}_1 = (2,1,1,5), \quad \vec{b}_2 = (1,-1,2,1).$$

Both $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ and $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ are bases of W. The matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ has $[\vec{b}_j]_{\mathcal{C}}$ as its jth column.

a) Find $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and $P_{\mathcal{B}\leftarrow\mathcal{C}}$.

b) A vector $\vec{x} \in W$ has $[\vec{x}]_{\mathcal{B}} = (1,-2)$. Find $[\vec{x}]_{\mathcal{C}}$ and $\vec{x}$ itself, and check your answer.

c) Find $[\vec{y}]_{\mathcal{B}}$ for $\vec{y} = (3,-3,6,3)$.

The Matrix of a Transformation Relative to Given Bases

3. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map $T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ 3x_1 + x_3)$, and take the bases

$$\mathcal{B} = \{(1,1,0),\ (0,1,1),\ (1,0,1)\} \text{ of } \mathbb{R}^3, \qquad \mathcal{C} = \{(1,1),\ (1,2)\} \text{ of } \mathbb{R}^2.$$

The matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ has $[T(\vec{b}_j)]_{\mathcal{C}}$ as its jth column.

a) Find $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$.

b) The vector $\vec{v}$ has $[\vec{v}]_{\mathcal{B}} = (1, 2, -1)$. Use your matrix to find $[T(\vec{v})]_{\mathcal{C}}$, then $T(\vec{v})$, and check by computing $\vec{v}$ and applying T directly.

4. Let V have basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and W have basis $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$. A linear map $T\colon V \to W$ has

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \end{bmatrix},$$

where the jth column is $[T(\vec{b}_j)]_{\mathcal{C}}$.

a) Write $T(\vec{b}_2)$ and $T(\vec{b}_1 - \vec{b}_3)$ in terms of $\vec{c}_1$ and $\vec{c}_2$.

b) Find $T(2\vec{b}_1 - \vec{b}_2 + \vec{b}_3)$.

c) Find a nonzero vector of $\ker T$, written in terms of $\vec{b}_1, \vec{b}_2, \vec{b}_3$.

Matrix Representations on Polynomial and Matrix Spaces

5. Let $T\colon P_2 \to P_3$ be the linear map $T(p)(x) = (x-2)\,p(x)$.

a) Find the matrix of T relative to the standard bases $\{1, x, x^2\}$ of P_2 and $\{1, x, x^2, x^3\}$ of P_3 (the jth column is the coordinate vector of the image of the jth basis polynomial).

b) Find the matrix of T relative to the bases $\{1,\ x-2,\ (x-2)^2\}$ of P_2 and $\{1,\ x-2,\ (x-2)^2,\ (x-2)^3\}$ of P_3.

The Change of Basis Formula for a Linear Operator

6. The linear operator $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ has standard matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \end{bmatrix}.$$

Let $\mathcal{B} = \{(1,0,0),\ (1,1,0),\ (1,1,1)\}$ and let P be the matrix with these vectors as columns.

a) Find P^{-1} and use $[T]_{\mathcal{B}} = P^{-1}AP$ to find $[T]_{\mathcal{B}}$.

b) Check the second column of your answer directly: compute $T(1,1,0)$ and confirm that it equals the combination of the vectors of $\mathcal{B}$ that the column prescribes.

7. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ with $\vec{b}_1 = (1,1)$ and $\vec{b}_2 = (1,2)$. A linear operator T on $\mathbb{R}^2$ satisfies $T(\vec{b}_1) = 2\vec{b}_1 - \vec{b}_2$ and $T(\vec{b}_2) = \vec{b}_1$.

a) Write down $[T]_{\mathcal{B}}$, whose columns are $[T(\vec{b}_1)]_{\mathcal{B}}$ and $[T(\vec{b}_2)]_{\mathcal{B}}$.

b) Use the change-of-basis formula to find the standard matrix A of T.

c) Find $T(3,5)$ with A, and again by writing $(3,5)$ in terms of $\mathcal{B}$.

Similar Matrices and Similarity Invariants

8. Two $n \times n$ matrices A and B are *similar* if $B = P^{-1}AP$ for some invertible P; similar matrices have the same trace, the same determinant and the same rank. For each pair, either name an invariant that shows A and B are *not* similar, or verify that $B = P^{-1}AP$ for the P given (check $AP = PB$).

a) $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

c) $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 2 & 6 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Synthesis — drawing on several topics in this unit

9. Let $T\colon P_2 \to P_2$ be the linear operator $T(p)(x) = x\,p'(x)$, let $\mathcal{E} = \{1, x, x^2\}$ and let $\mathcal{C} = \{1,\ 1+x,\ 1+x+x^2\}$. For an operator, $[T]_{\mathcal{B}}$ has $[T(b_j)]_{\mathcal{B}}$ as its jth column.

a) Find $[T]_{\mathcal{E}}$.

b) Find $P = P_{\mathcal{E}\leftarrow\mathcal{C}}$ (columns $[c_j]_{\mathcal{E}}$) and P^{-1}, and use $[T]_{\mathcal{C}} = P^{-1}[T]_{\mathcal{E}}P$ to find $[T]_{\mathcal{C}}$.

c) Check the third column of $[T]_{\mathcal{C}}$ by computing $T(1+x+x^2)$ directly.

d) Name two numbers that $[T]_{\mathcal{E}}$ and $[T]_{\mathcal{C}}$ must share, and confirm that they do.