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Eigenvalues and Diagonalization

Name: ______________________________ Date: ______________

Concepts

- Eigenvalues and Eigenvectors from the Definition
- The Characteristic Polynomial
- Eigenspaces and Their Dimensions
- Diagonalizing a Matrix
- Deciding Whether a Matrix Is Diagonalizable
- Powers of a Diagonalizable Matrix (theory)
- Complex Eigenvalues of a Real Matrix (theory)

Eigenvalues and Eigenvectors from the Definition

1. Let

$$A = \begin{bmatrix} 3 & -4 & -4 & -4 \\ 1 & 4 & 2 & 2 \\ 0 & -3 & -1 & -3 \\ -1 & -2 & -2 & 0 \end{bmatrix}.$$

a) Working from the definition only (no characteristic polynomial), decide which of the following are eigenvectors of A, and give the eigenvalue of each one that is:

$$\vec{u} = (1,1,0,-1), \qquad \vec{v} = (0,-1,-1,2), \qquad \vec{w} = (1,0,0,0), \qquad \vec{0} = (0,0,0,0).$$

b) Decide whether $\lambda = 1$ is an eigenvalue of A.

2. Let V be the vector space of functions $f\colon \mathbb{R} \to \mathbb{R}$ that have derivatives of every order, and let $D\colon V \to V$ be differentiation, $D(f) = f'$. An eigenvector of a linear operator T is a nonzero f with $T(f) = \lambda f$ for some real λ. For each of

$$f_1(x) = e^{-4x}, \qquad f_2(x) = xe^{x}, \qquad f_3(x) = 5, \qquad f_4(x) = \cos 2x,$$

decide whether it is an eigenvector of D, and whether it is an eigenvector of $D^2 = D \circ D$. Give the eigenvalue whenever it is one.

The Characteristic Polynomial

3. The characteristic polynomial of a square matrix A is $p(\lambda) = \det(A - \lambda I)$. Let

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}.$$

a) Find $p(\lambda)$ in factored form.

b) List the eigenvalues of A with their algebraic multiplicities.

c) Check that the eigenvalues, each counted as often as its multiplicity, add up to $\mathrm{tr}(A)$ and multiply to $\det(A)$.

4. Let

$$A = \begin{bmatrix} 1 & 2 & 5 & -1 \\ 3 & 2 & 0 & 4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Find the characteristic polynomial $\det(A - \lambda I)$ in factored form and the eigenvalues of A. Check your eigenvalues against $\mathrm{tr}(A)$.

Eigenspaces and Their Dimensions

5. Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 3 & 2 & 0 & 3 \\ -3 & 1 & 2 & -2 \\ -3 & -1 & 0 & -2 \end{bmatrix}.$$

Given that $\lambda = 2$ is an eigenvalue of A:

a) find a basis for the eigenspace E_2 and state $\dim E_2$;

b) write E_2 in set-builder notation.

6. Let $T\colon M_{2\times 2} \to M_{2\times 2}$ be the linear map $T(X) = X^T$.

a) Show that if $T(X) = \lambda X$ for a nonzero X, then $\lambda = 1$ or $\lambda = -1$. (Apply T twice.)

b) Describe the eigenspaces E_1 and E_{-1} in set-builder notation, and give a basis and the dimension of each.

Diagonalizing a Matrix

7. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find an invertible matrix P and a diagonal matrix D with $A = PDP^{-1}$. Check your answer by verifying $AP = PD$.

8. Let $T\colon P_2 \to P_2$ be the linear map $T(p(x)) = p(2x+1)$, where P_2 is the space of polynomials of degree at most 2.

a) Find the matrix $[T]_{\mathcal{B}}$ of T relative to the basis $\mathcal{B} = \{1, x, x^2\}$, and its eigenvalues.

b) Find a basis $\mathcal{C}$ of P_2 consisting of eigenvectors of T, and write down $[T]_{\mathcal{C}}$.

Deciding Whether a Matrix Is Diagonalizable

9. Each matrix below is given with its characteristic polynomial $\det(M - \lambda I)$. Decide whether it is diagonalizable, and justify.

a) $A = \begin{bmatrix} 4 & 5 & 0 & -5 \\ -3 & -4 & -2 & 5 \\ 0 & 4 & 3 & -4 \\ -3 & -3 & -2 & 4 \end{bmatrix}$, $\quad (\lambda - 1)(\lambda + 1)(\lambda - 3)(\lambda - 4)$.

b) $B = \begin{bmatrix} 2 & 0 & 1 \\ -3 & 5 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $\quad -(\lambda - 2)^2(\lambda - 5)$.

c) $C = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 5 & 3 \\ 0 & 0 & 2 \end{bmatrix}$, $\quad -(\lambda - 2)^2(\lambda - 5)$.

10. For which real numbers k is

$$A = \begin{bmatrix} 2 & k & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

diagonalizable? Justify.

Powers of a Diagonalizable Matrix

11. Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$.

a) Diagonalize A.

b) Find a formula for A^n, $n \geq 1$, and use it to write down A^{10}.

Complex Eigenvalues of a Real Matrix

12. Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$, and work with real scalars.

a) Find the roots of the characteristic polynomial $\det(A - \lambda I)$.

b) Does A have a real eigenvector? Is A diagonalizable over $\mathbb{R}$?

c) Is there a line through the origin in $\mathbb{R}^2$ that multiplication by A maps into itself? Explain.

Synthesis — drawing on several topics in this unit

13. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

a) Find the characteristic polynomial $\det(A - \lambda I)$, and the eigenvalues with their algebraic multiplicities.

b) Find a basis for each eigenspace.

c) Decide whether A is diagonalizable; if it is, give P and D with $A = PDP^{-1}$.

d) Use (c) to find A^{50}.