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Inner Products and Orthogonality

Name: ____ Date: __________________

Concepts

- Inner Product, Norm and Distance in n-Space
- Inner Products on Polynomial and Function Spaces (theory)
- Orthogonal and Orthonormal Sets
- Coordinates Relative to an Orthogonal Basis
- The Orthogonal Complement of a Subspace (theory)

Inner Product, Norm and Distance in n-Space

1. In $\mathbb{R}^5$ with the dot product, let $\vec{u} = (1, -2, 0, 2, 4)$ and $\vec{v} = (2, 1, -2, 0, 4)$.

a) Find $\vec{u} \cdot \vec{v}$, $\|\vec{u}\|$ and $\|\vec{v}\|$.

b) Find the distance $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$.

c) Find the unit vector in the direction of $\vec{v}$.

d) Find $\cos\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$, and say whether θ is acute, right or obtuse.

2. On $\mathbb{R}^4$ define

$$\langle \vec{x}, \vec{y} \rangle = 2x_1y_1 + x_2y_2 + 3x_3y_3 + x_4y_4,$$

where $\vec{x} = (x_1, x_2, x_3, x_4)$ and $\vec{y} = (y_1, y_2, y_3, y_4)$. Norm and distance are $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$ and $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$, computed with this inner product. Let $\vec{x} = (1, -1, 1, 2)$ and $\vec{y} = (0, 3, 1, -1)$.

a) Explain why $\langle \vec{x}, \vec{x} \rangle > 0$ for every $\vec{x} \neq \vec{0}$.

b) Find $\langle \vec{x}, \vec{y} \rangle$, $\|\vec{x}\|$, $\|\vec{y}\|$ and $d(\vec{x}, \vec{y})$. Compare $\langle \vec{x}, \vec{y} \rangle$ and $d(\vec{x}, \vec{y})$ with the values the dot product gives.

c) Show that $\vec{x}$ and $\vec{z} = (1, 2, 0, 0)$ are orthogonal for this inner product but not for the dot product.

Inner Products on Polynomial and Function Spaces

3. On P_2, the polynomials of degree at most 2, use the inner product

$$\langle p, q\rangle = \int_{-1}^{1} p(x)\, q(x)\, dx,$$

with $\|p\| = \sqrt{\langle p, p\rangle}$ and $d(p, q) = \|p - q\|$. Let $p(x) = 1 + 2x$ and $q(x) = x - x^2$.

a) Find $\langle p, q\rangle$.

b) Find $\|p\|$ and $\|q\|$.

c) Find the distance $d(p, q)$.

d) Find the value of the constant a for which $r(x) = a + x^2$ is orthogonal to q.

Orthogonal and Orthonormal Sets

4. In $\mathbb{R}^4$ with the dot product, let $\vec{v}_1 = (1, 2, 0, -1)$, $\vec{v}_2 = (-2, 1, -2, 0)$ and $\vec{v}_3 = (1, 0, -1, 1)$.

a) Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set.

b) Turn it into an orthonormal set.

c) Is the set linearly independent? Is it a basis of $\mathbb{R}^4$? Justify both answers without row reducing.

5. On $M_{2\times2}$ use the inner product $\langle A, B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$, the sum of the products of corresponding entries. Let

$$A_1 = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}.$$

a) Show that $\{A_1, A_2, A_3\}$ is an orthogonal set.

b) Turn it into an orthonormal set.

c) Is $\{A_1, A_2, I_2\}$ an orthogonal set?

Coordinates Relative to an Orthogonal Basis

6. In $\mathbb{R}^4$ with the dot product, let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\}$ with

$$\vec{b}_1 = (3, 1, 1, -1), \quad \vec{b}_2 = (-1, -1, 1, -3), \quad \vec{b}_3 = (-1, 2, 1, 0), \quad \vec{b}_4 = (0, -1, 2, 1).$$

a) Show that $\mathcal{B}$ is an orthogonal basis of $\mathbb{R}^4$.

b) Find $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = (5, 1, 2, -3)$ without solving a linear system.

7. On P_2 use the inner product $\langle p, q\rangle = p(0)\,q(0)+p(1)\,q(1)+p(2)\,q(2)$. Let $\mathcal{B} = \{1,\ x-1,\ 3x^2-6x+1\}$.

a) Show that $\mathcal{B}$ is an orthogonal basis of P_2 for this inner product.

b) Find the coordinates of $p(x) = 2x^2 - x + 3$ relative to $\mathcal{B}$ by computing inner products, and check your answer.

The Orthogonal Complement of a Subspace

8. In $\mathbb{R}^5$ with the dot product, let $W = \operatorname{span}\{(1,0,2,-1,1),\ (0,1,-1,2,0)\}$. The orthogonal complement is $W^\perp = \{\vec{x} \in \mathbb{R}^5 \mid \vec{x} \cdot \vec{w} = 0 \text{ for every } \vec{w} \in W\}$. Find a basis for $W^\perp$, and check that $\dim W + \dim W^\perp = 5$.

Synthesis — drawing on several topics in this unit

9. In $\mathbb{R}^4$ with the dot product, let $\vec{v}_1 = (2,1,-1,1)$, $\vec{v}_2 = (-1,-1,-2,1)$, $\vec{v}_3 = (0,-1,2,3)$ and $W = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

a) Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set, and find $\dim W$.

b) Find a basis for $W^\perp$, the set of vectors orthogonal to every vector of W.

c) Let $\vec{u}$ be the unit vector in $W^\perp$ whose last component is positive. Find the coordinates of $\vec{x} = (3,0,1,2)$ relative to the orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{u}\}$ of $\mathbb{R}^4$.