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Projections, Least Squares and Symmetric Matrices

Name: ______________________________ Date: ______________

Concepts

- Orthogonal Projection onto a Subspace
- The Gram-Schmidt Process
- Least Squares Solutions and the Normal Equations
- Fitting a Line or Curve by Least Squares (theory)
- Orthogonal Matrices (theory)
- Orthogonal Diagonalization of a Symmetric Matrix

Orthogonal Projection onto a Subspace

1. In $\mathbb{R}^4$ with the dot product, let $W = \operatorname{span}\{\vec{u}_1, \vec{u}_2\}$, where

$$\vec{u}_1 = (1, 2, 0, 1), \qquad \vec{u}_2 = (2, -1, 1, 0), \qquad \text{and let} \qquad \vec{y} = (1, 6, -2, 5).$$

a) Check that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set.

b) Find $\operatorname{proj}_W \vec{y}$.

c) Write $\vec{y} = \hat{\vec{y}} + \vec{z}$ with $\hat{\vec{y}} \in W$ and $\vec{z} \in W^{\perp}$, and verify that $\vec{z}$ is orthogonal to W.

d) Find the distance from $\vec{y}$ to W.

2. On P_3, the polynomials of degree at most 3, use the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx.$$

Let $W = \text{span}\{1,\, x\}$ and $p(x) = x^3 + x^2$.

a) Show that 1 and x are orthogonal for this inner product.

b) Find $\text{proj}_W\, p$, the polynomial of degree at most 1 closest to p for this inner product.

c) Verify that $p - \text{proj}_W\, p$ is orthogonal to both 1 and x.

The Gram-Schmidt Process

3. In $\mathbb{R}^4$ with the dot product, let

$$\vec{x}_1 = (1,1,0,1), \qquad \vec{x}_2 = (2,0,1,1), \qquad \vec{x}_3 = (1,2,-2,0).$$

a) Apply the Gram-Schmidt process to $\vec{x}_1, \vec{x}_2, \vec{x}_3$, in that order, to obtain an orthogonal basis of $W = \operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$.

b) Normalize it to an orthonormal basis of W.

4. On P_2, the polynomials of degree at most 2, use the inner product

$$\langle p, q\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2).$$

a) Apply the Gram-Schmidt process to the basis $\{1, x, x^2\}$, in that order.

b) Normalize the result to an orthonormal basis of P_2 for this inner product.

Least Squares Solutions and the Normal Equations

5. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} -1 \\ -1 \\ -2 \\ 3 \end{bmatrix}.$$

a) Show that $A\vec{x} = \vec{b}$ has no solution.

b) Write the normal equations $A^T A\vec{x} = A^T\vec{b}$ and solve them for the least-squares solution $\hat{\vec{x}}$.

c) Find the least-squares error $\|\vec{b} - A\hat{\vec{x}}\|$.

6. Let

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 5 \end{bmatrix}.$$

A vector $\vec{x}$ is a least-squares solution of $A\vec{x} = \vec{b}$ exactly when $A^T(\vec{b} - A\vec{x}) = \vec{0}$. Use this test to decide which of $\vec{p} = (5, 2)$ and $\vec{q} = (2, -2)$ is a least-squares solution, then compare $\|\vec{b} - A\vec{p}\|$ and $\|\vec{b} - A\vec{q}\|$.

Fitting a Line or Curve by Least Squares

7. The depth of water in a test channel, in centimetres, is recorded x hours after a gate is opened:

$$(x, y) = (0, 1),\ (1, 3),\ (2, 4),\ (3, 4).$$

a) Write the system $X\vec{\beta} = \vec{y}$ that a line $y = \beta_0 + \beta_1 x$ through all four points would have to satisfy.

b) Solve the normal equations for the least-squares line.

c) Find the sum of the squared residuals.

Orthogonal Matrices

8. A square matrix Q is *orthogonal* when $Q^T Q = I$. Decide which of the following matrices are orthogonal, and give the inverse of each one that is.

$$M_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad M_3 = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 2 & -2 \\ 1 & 2 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Orthogonal Diagonalization of a Symmetric Matrix

9. Orthogonally diagonalize

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix}:$$

find an orthogonal matrix P and a diagonal matrix D with $A = PDP^T$.

10. The symmetric matrix

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

has eigenvalues -2 and 2 only. Find a basis of each eigenspace, and then an orthogonal matrix P and a diagonal matrix D with $A = PDP^T$.

Synthesis — drawing on several topics in this unit

11. A technician codes four equally spaced times as $x = -3, -1, 1, 3$ and reads a voltage y, in millivolts, at each: $y = 1,\ 2,\ 4,\ 7$. The least-squares line $y = \beta_0 + \beta_1 x$ is wanted.

a) Write the design matrix X and show that its two columns are orthogonal.

b) Use orthogonal projection onto $\mathrm{Col}(X)$ to find the fitted values $\hat{\vec{y}}$, and read off β_0 and β_1.

c) Confirm the line with the normal equations, and find the sum of the squared residuals.

Want the harder problems and the worked solutions?

The *Challenge Problems* set and the *Answer Keys* for Projections, Least Squares and Symmetric Matrices — with the reasoning written out in full — are available at **tutorinmontreal.ca**.