University Introductory Statistics · Practice Worksheet · tutorinmontreal.ca

Continuous Distributions and the Normal Model

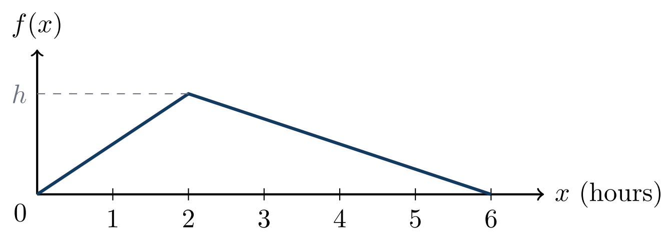
Name: ____ Date: __________________

Concepts

- Density Functions and Probability as Area (theory)
- Mean and Variance of a Continuous Random Variable (theory)
- The Uniform and Exponential Distributions (theory)
- The Standard Normal Table and Standardizing
- Percentiles of a Normal Distribution
- The Normal Approximation to the Binomial

Density Functions and Probability as Area

1. The time X, in hours, that a technician needs to complete a repair has a density f whose graph is a triangle: f rises in a straight line from 0 at $x = 0$ to a maximum height h at $x = 2$, then falls in a straight line to 0 at $x = 6$, and $f(x) = 0$ outside $[0, 6]$.



a) Find h.

b) Find $P(X \leq 1)$ and $P(X > 4)$.

c) Find $P(1 \leq X \leq 4)$.

d) What is $P(X = 2)$? Explain in one sentence.

Mean and Variance of a Continuous Random Variable

2. The time X, in minutes, that a barista takes to prepare an order has a triangular density: it rises in a straight line from 0 at $x = 2$ to a peak at $x = 5$, then falls in a straight line to 0 at $x = 8$, and is 0 outside $[2, 8]$.

a) Find the height of the peak.

b) Find $E[X]$, and justify your answer without any formula.

c) For a triangular density on $[a, b]$ with its peak at c,

$$\mathrm{Var}(X) = \frac{a^2 + b^2 + c^2 - ab - ac - bc}{18}.$$

Find $\mathrm{Var}(X)$ and $\mathrm{SD}(X)$.

d) Find $P(X > 7)$.

The Uniform and Exponential Distributions

3.

a) A ferry leaves every 20 minutes, and a passenger who ignores the timetable arrives at a random moment, so the wait W, in minutes, is uniform on $[0, 20]$. Find $P(W > 15)$ and $P(5 < W < 12)$.

b) For a uniform variable on $[a, b]$, $E[W] = \dfrac{a+b}{2}$ and $\text{Var}(W) = \dfrac{(b-a)^2}{12}$. Find the mean and standard deviation of the wait.

c) At a bike-share dock, the time T, in minutes, between two returns is exponential with mean 4 minutes, so $P(T \le t) = 1 - e^{-t/4}$ for $t \ge 0$. Find $P(T > 8)$ and $P(2 < T \le 6)$, exactly and to four decimals. Use $e^{-0.5} = 0.6065$, $e^{-1.5} = 0.2231$, $e^{-2} = 0.1353$.

The Standard Normal Table and Standardizing

4. Let Z be a standard normal variable. The table available gives $P(Z < z)$ for $z \ge 0$ only:

$$P(Z < 0.62) = 0.7324, \quad P(Z < 0.84) = 0.7995, \quad P(Z < 1.37) = 0.9147, \quad P(Z < 1.50) = 0.9332.$$

Find:

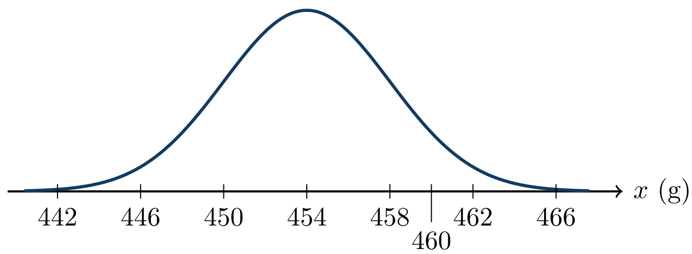
a) $P(Z < 1.37)$;

b) $P(Z > -0.84)$;

c) $P(-1.50 < Z < 0.62)$;

d) $P(|Z| > 1.37)$.

5. We write $X \sim N(\mu, \sigma^2)$ for a normal variable with mean μ and *variance* σ^2 (so the second number is the variance, not the standard deviation). The mass X, in grams, of coffee in a bag labelled 454 g is $X \sim N(454, 16)$. The curve below is its density, with the mean and the points one, two and three standard deviations either side marked.



Use $P(Z < 1.00) = 0.8413$ and $P(Z < 1.50) = 0.9332$.

a) Find $P(X < 448)$.

b) Find $P(450 < X < 460)$, and shade that region on the curve.

c) Using the 68–95–99.7 rule, about what percentage of bags hold between 446 and 462 g? Between 442 and 466 g?

Percentiles of a Normal Distribution

6. We write $N(\mu, \sigma^2)$ with the variance second. The battery life X, in hours, of a tablet is $N(11,\ 2.25)$. Use whichever of these you need:

$$P(Z < 1.2816) = 0.9000, \qquad P(Z < 1.6449) = 0.9500, \qquad P(Z < 1.9600) = 0.9750.$$

a) Find the 90th percentile of battery life.

b) Find the battery life that only 5% of tablets fall short of.

c) Find the interval, symmetric about the mean, that contains the middle 80% of battery lives.

7. We write $N(\mu, \sigma^2)$ with the variance second. A courier's delivery time T, in hours, is $N(\mu, 16)$, and the courier can shift μ by changing its routing. Use whichever of these you need:

$$P(Z < 2.0537) = 0.9800, \qquad P(Z < 2.3263) = 0.9900.$$

a) Find the largest mean μ for which only 1% of deliveries take longer than 48 hours.

b) Suppose instead the mean is fixed at 40 hours and the courier works on reliability. What standard deviation would give the same guarantee, $P(T > 48) = 0.01$?

The Normal Approximation to the Binomial

8. A survey firm places 150 calls, and each call is answered independently with probability 0.40. Let X be the number answered. Take the normal approximation to be acceptable when $np \geq 10$ and $n(1-p) \geq 10$. Use $P(Z < 0.75) = 0.7734$ and $P(Z < 1.75) = 0.9599$.

a) Find the mean and standard deviation of X, and check that the normal approximation is acceptable.

b) Using the continuity correction, approximate $P(X \geq 71)$.

c) Using the continuity correction, approximate $P(56 \leq X \leq 64)$.

9.

a) Take the normal approximation to $\text{Bin}(n, p)$ to be acceptable when $np \geq 10$ and $n(1-p) \geq 10$. For each of the following, decide whether it is acceptable: Bin(40, 0.1); Bin(400, 0.03); Bin(60, 0.95).

b) X is binomial and Y is the normal variable approximating it. Rewrite each event about X as an event about Y, using the continuity correction: $X < 30$; $X \leq 30$; $X = 30$; $X > 30$; $25 \leq X < 30$.

Synthesis — drawing on several topics in this unit

10. We write $N(\mu, \sigma^2)$ with the variance second. The volume of paint in a tin labelled 1 litre is $N(1000, 64)$ millilitres, and a tin is *underfilled* if it holds less than 988 mL. Tins are filled independently. Take the normal approximation to the binomial to be acceptable when $np \geq 10$ and $n(1-p) \geq 10$. Use whichever of these you need: $P(Z < 1.09) = 0.8621$, $P(Z < 1.18) = 0.8810$, $P(Z < 1.50) = 0.9332$.

a) Find the probability p that a tin is underfilled.

b) A shipment holds 500 tins. Using the continuity correction, approximate the probability that 40 or more of them are underfilled.